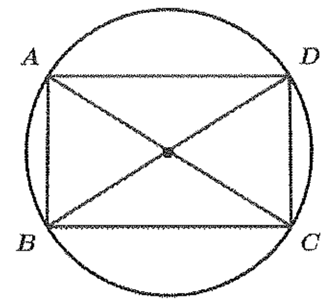
Homework Helpers

Geometry
Module 5

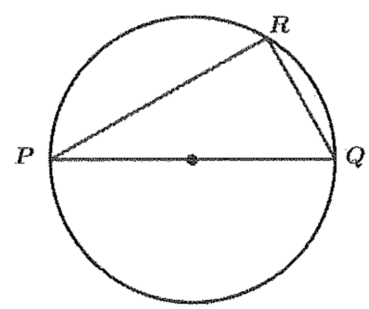
Lesson 1: Thales' Theorem

1. Geoffrey claims that if four points on a circle form a rectangle, then the diagonals of the rectangle are diameters of the circle. Is he correct?



Geoffrey is correct. Let A, B, C, and D be four points on a circle so that $ABCD$ is a rectangle with diagonals $\overline{AC}$ and $\overline{BD}$. The converse of Thales' theorem states that since $\angle ABC$ is a right angle, points A, B, and C lie on a circle with diameter $\overline{AC}$. Similarly, since $\angle BCD$ is a right angle, points B, C, and D lie on a circle with diameter $\overline{BD}$. Therefore, the diagonals of the rectangle are diameters of the circle.

2. In the figure to the right, $\overline{PQ}$ is the diameter of a circle with radius 20 cm. If $\overline{PR}$ is 35 cm long, what is the length of $\overline{QR}$?



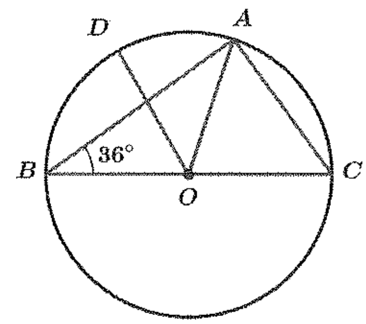
Because $\overline{PQ}$ is a diameter of the circle, Thales' theorem states that $\angle R$ is a right angle, and $\triangle PRQ$ is a right triangle with hypotenuse length of 40 cm and one leg of length 35 cm. Let $c = 40$ and $b = 35$. Then

$$a^2 + b^2 = c^2$$
$$a^2 = c^2 - b^2$$
$$a^2 = 40^2 - 35^2$$
$$a^2 = 1600 - 1225$$
$$a^2 = 375$$
$$a = \sqrt{375}$$
$$a \approx 19.4$$

> Since $\triangle PRQ$ is a right triangle, I can use the Pythagorean theorem to find the missing leg length.

Segment QR is approximately 19.4 centimeters long.

3. A circle with center O has diameter $\overline{BC}$.



a. Point A is a point on the circle so that $m\angle ABO = 36°$. Find $m\angle OAC$.

$m\angle BAO = m\angle ABO = 36°$

$m\angle BAC = 90°$

$m\angle OAC = m\angle BAC - m\angle BAO$
$m\angle OAC = 90° - 36°$
$m\angle OAC = 54°$

Since $\overline{OA}$ and $\overline{OB}$ are both radii of the circle, I know that $\triangle ABO$ is isosceles.

Since $\overline{BC}$ is a diameter of the circle, I know that $\angle BAC$ is a right angle.

b. Point D is a point on the circle as shown so that $m\angle DOA::m\angle AOC$ is 2: 3. Find $m\angle DOA$.

$m\angle OCA = m\angle OAC = 54°$

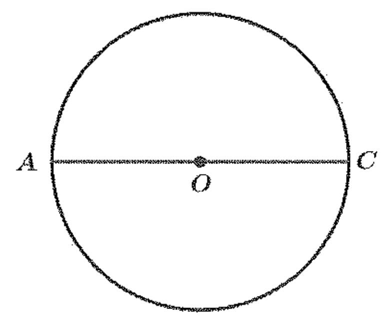
$m\angle AOC + m\angle OCA + m\angle OAC = 180°$
$m\angle AOC + 54° + 54° = 180°$
$m\angle AOC = 180° - 54° - 54°$
$m\angle AOC = 72°$

Since $\overline{OA}$ and $\overline{OC}$ are both radii of the circle, I know that $\triangle AOC$ is isosceles.

Because $m\angle DOA$: $m\angle AOC$ *is* 2: 3, $m\angle DOA = \frac{2}{3} m\angle AOC$.

Therefore, $m\angle DOA = \frac{2}{3}(72°) = 48°$.

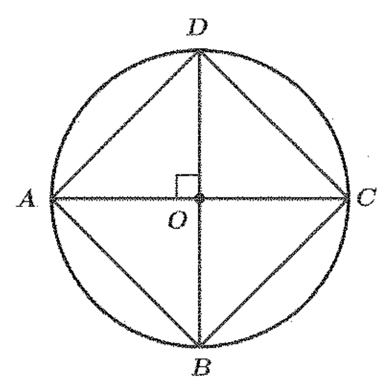
4. Devise a method for inscribing a square in a given circle with center O and diameter $\overline{AC}$. Prove that your method produces a square.



Given a circle with center O and diameter $\overline{AC}$, *draw a line perpendicular to* $\overleftrightarrow{AC}$ *at* O. *This line will intersect the circle at two points,* B *and* D. *Draw* $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, *and* $\overline{DA}$ *to form quadrilateral* $ABCD$.

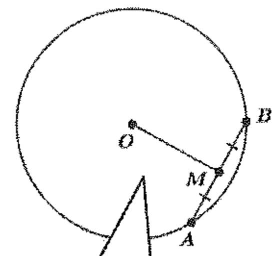
Since $\overline{AC}$ *is a diameter of the circle,* $\angle ADC$ *and* $\angle ABC$ *are right angles by Thales' theorem. Since* $\overline{BD}$ *is also a diameter of the circle,* $\angle BAD$ *and* $\angle BCD$ *are also right angles.*

Since $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, *and* $\overline{OD}$ *are all radii of the same circle,* $\overline{OA} \cong \overline{OB} \cong \overline{OC} \cong \overline{OD}$. *Since* $\overleftrightarrow{AC} \perp \overleftrightarrow{BD}$, *the right angles formed by their intersection are all equal in meaure:* $\angle AOD \cong \angle DOC \cong \angle COB \cong \angle BOA$. *Then,* $\triangle AOD \cong \triangle DOC \cong \triangle COB \cong \triangle BOA$ *by SAS congruence. Because corresponding parts of congruent triangles are congruent,* $\overline{AD} \cong \overline{CD} \cong \overline{CB} \cong \overline{BA}$. *Thus, quadrilateral* $ABCD$ *has four right angles and four congruent sides, and* $ABCD$ *is a square.*



Lesson 2: Circles, Chords, Diameters, and Their Relationships

1. In the figure, O is the center of a circle with radius 20, $AB = 20$, and M is the midpoint of $\overline{AB}$. Find OM.

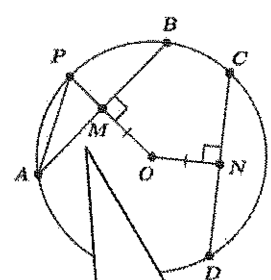


If I draw in the radius $\overline{OA}$, I can create a triangle that I can show is a right triangle.

$AB = 20$; M *is the midpoint of* $\overline{AB}$.	*Given*
$AM = 10$	*Definition of midpoint*
$\overline{OM} \perp \overline{AB}$	*If a diameter bisects a chord, then it is perpendicular to the chord.*
$\triangle OMA$ *is a right triangle.*	*Definition of right triangle*
$OA = 20$	*All radii in a circle have the same length.*
$(OM)^2 + (AM)^2 = (OA)^2$	*Pythagorean theorem*
$(OM)^2 + 10^2 = 20^2$	*Substitution*

$$OM = \sqrt{20^2 - 10^2} = \sqrt{300} = 10\sqrt{3}$$

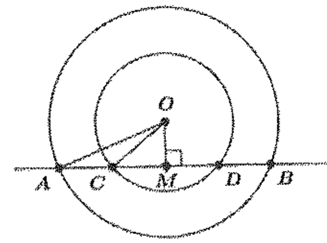
2. In the figure, $\overline{OM} \perp \overline{AB}$ and $\overline{ON} \perp \overline{CD}$, O is the center of the circle with radius 20, $OM = ON = 12$, and $CD = 32$. Find AP.



I can see that $\triangle PMA$ is a right triangle, so once I find the lengths of the legs $\overline{PM}$ and $\overline{AM}$, I can use the Pythagorean theorem to find the length of the hypotenuse $\overline{AP}$.

$\overline{OM} \perp \overline{AB}$ *and* $\overline{ON} \perp \overline{CD}$, $OM = ON = 12$, $CD = 32$	*Given*
$\triangle ONC$ *and* $\triangle OMA$ *are right triangles.*	*Definition of right triangle*
$OA = 20$; $OP = 20$; $OC = 20$	*All radii in a circle have the same length.*
$\triangle ONC \cong \triangle OMA$	*Hypotenuse-leg congruence criterion*
$\overrightarrow{OM}$ *bisects* $\overline{AB}$ *and* $\overrightarrow{ON}$ *bisects* $\overline{CD}$.	*If a diameter is perpendicular to a chord, then it bisects the chord.*
$CN = \frac{1}{2}(CD) = \frac{1}{2}(32) = 16$	*Definition of bisect*
$AM = CN = 16$	*Corresponding parts of congruent triangles have equal measure.*
$PM = OP - OM = 20 - 12 = 8$	*Substitution*
$\triangle PMA$ *is a right triangle.*	*Definition of right triangle*
$(AP)^2 = (PM)^2 + (AM)^2$	*Pythagorean theorem*
$(AP)^2 = 8^2 + 16^2$	*Substitution*
$(AP)^2 = 320$	
$AP = 8\sqrt{5}$	

3. In the figure, O is the center of both circles, $\overline{OM} \perp \overline{AB}$, the large circle has radius 39, the small circle has radius 17, and $AB = 72$. Find CD.

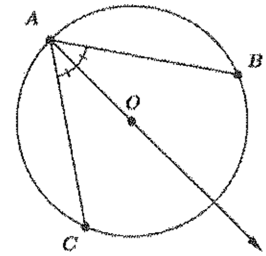


$\overline{OM} \perp \overline{AB}$ *and* $AB = 72$	*Given*
M *is the midpoint of* $\overline{AB}$; M *is the midpoint of* $\overline{CD}$.	*If a diameter is perpendicular to a chord, then it bisects the chord.*
$AM = \frac{1}{2}(AB) = \frac{1}{2}(72) = 36$	*Definition of midpoint*
$\triangle OMA$ *is a right triangle.*	*Definition of right triangle*
$OA = 39$	*All radii in a circle have the same length.*
$(OM)^2 + (AM)^2 = (OA)^2$	*Pythagorean theorem*
$(OM)^2 + 36^2 = 39^2$	*Substitution*
$OM = \sqrt{39^2 - 36^2} = \sqrt{225} = 15$	
$\triangle OMC$ *is a right triangle.*	*Definition of right triangle*
$OC = 17$	*All radii in a circle have the same length.*
$(OM)^2 + (CM)^2 = (OC)^2$	*Pythagorean theorem*
$15^2 + (CM)^2 = 17^2$	*Substitution*
$CM = \sqrt{17^2 - 15^2} = \sqrt{64} = 8$	
$CD = CM + MD$	*Segment addition*
$CM = MD$	*Definition of midpoint*
$CD = 2(CM) = 2(8) = 16$	*Substitution*

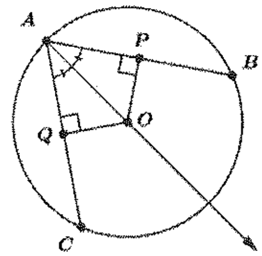
If I can find the height of $\triangle OMC$, I can use the Pythagorean theorem to find the length of leg $\overline{CM}$.

4. In the figure, O is the center of the circle and $\overrightarrow{AO}$ bisects $\angle BAC$. Prove that $AB = AC$.

If I draw perpendicular segments from the center O to the chords $\overline{AB}$ and $\overline{AC}$, then I create right triangles that I can prove are congruent.



Construct perpendicular segments to $\overline{AB}$ and $\overline{AC}$ through O. Let these intersect $\overline{AB}$ at P and $\overline{AC}$ at Q.

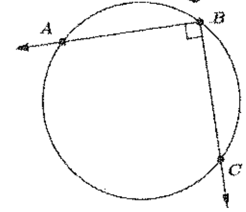


$\overline{AO} \cong \overline{AO}$	***Reflexive property***
$\angle CAO \cong \angle BAO$	***Definition of angle bisector***
$\angle OQA$, $\angle OPA$ ***are right angles***	***Perpendicular lines intersect to form right angles***
$\angle OQA \cong \angle OPA$	***Right angles are congruent.***
$\triangle OQA \cong \triangle OPA$	***AAS triangle congruence***
$\overline{OP} \cong \overline{OQ}$	***Corresponding parts of congruent triangles are congruent.***
$\overline{AB} \cong \overline{AC}$	***If the center of a circle is equidistant from two chords, then the chords are congruent.***
$AB = AC$	***Meaning of congruence***

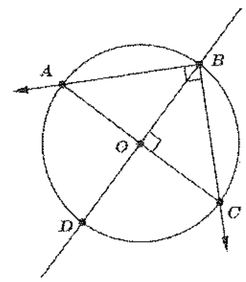
Lesson 3: Rectangles Inscribed in Circles

I can construct this right angle using my compass and straightedge.

1. Describe how to inscribe a square in a given circle.

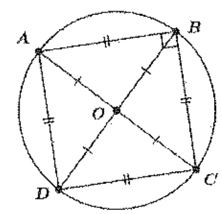


Construct a right angle inscribed in the circle at point B. Let A and C be the two points where the rays of the right angle intersect the circle.



Because $\angle ABC$ is a right angle, $\overline{AC}$ is a diameter of the circle. Let O be the midpoint of $\overline{AC}$; then O is the center of the circle. Construct a perpendicular line to $\overline{AC}$ at O; this will intersect the circle at point B and a new point D.

This is only one of many ways to construct an inscribed square.



Since $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$ are all radii of the circle, $\overline{OA} \cong \overline{OB} \cong \overline{OC} \cong \overline{OD}$. Therefore, $\triangle AOB \cong \triangle BOC \cong \triangle COD \cong \triangle DOA$ by SAS congruence, so $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$ because corresponding parts of congruent triangles are congruent. Then $ABCD$ is a rhombus with a right angle, so it is a square.

2. Let A and B be two points on a circle.

   a. Under what conditions can a rectangle with side $\overline{AB}$ be inscribed in the circle? Explain the process for constructing the rectangle.

   As long as $\overline{AB}$ is not a diameter of the circle, a rectangle with side $\overline{AB}$ can be inscribed in the circle.

   If $\overline{AB}$ is not a diameter of the circle, construct a perpendicular line to $\overline{AB}$ at B. This line will intersect the circle at point C. Since $\angle ABC$ is a right angle, $\overline{AC}$ is a diameter of the circle. Rotate $\triangle ABC$ by $180°$ about the origin, and let D be the image of B under this rotation. Then $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$ since rotation preserves length, and all four angles are right angles.

   This is the method used in the lesson, but there are other ways to construct a rectangle with side $\overline{AB}$.

   Thus, $ABCD$ is a rectangle.

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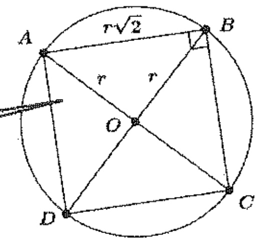
b. Under what conditions can a square with side $\overline{AB}$ be inscribed in the circle? Explain the process for constructing the square.

An inscribed square with side $\overline{AB}$ can only be constructed if $AB = r\sqrt{2}$, where r is the radius of the circle. If this condition is met, then the construction outlined in part (a) will produce a square.

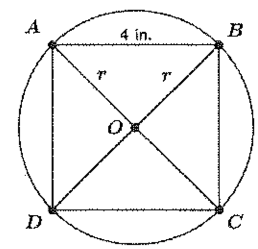


Applying the Pythagorean theorem to $\triangle AOB$, I know that $(OA)^2 + (OB)^2 = (AB)^2$.

3. Suppose that a square with side length 4 inches is inscribed in a circle. What is the radius of the circle?

Let r be the radius of the circle in which the square with side length 4 inches is inscribed.

$$r^2 + r^2 = 4^2$$
$$2r^2 = 16$$
$$r^2 = 8$$
$$r = \sqrt{8}$$

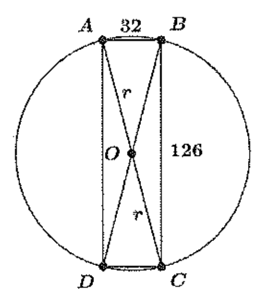


The radius of the circle is then $2\sqrt{2}$ inches.

4. Suppose that a rectangle with side lengths 32 cm and 126 cm is inscribed in a circle. What is the radius of the circle?

Suppose that $ABCD$ is inscribed in a circle of radius r, $AB = 32$, and $BC = 126$. Then diagonal $\overline{AC}$ has length $2r$, and $\triangle ABC$ is a right triangle. Then by the Pythagorean theorem,

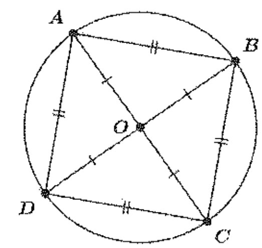
$$(AC)^2 = (AB)^2 + (BC)^2$$
$$(2r)^2 = 32^2 + 126^2$$
$$4r^2 = 1024 + 15876$$
$$r^2 = 4225$$
$$r = 65.$$



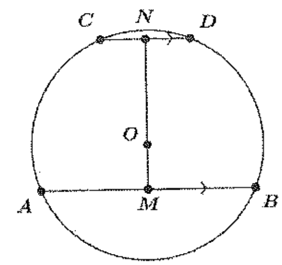
The radius of the circle is 65 cm.

5. Is it possible to inscribe a rhombus that is not a square in a circle? Explain how you know.

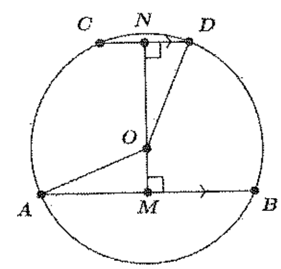
This is not possible. Suppose that a rhombus $ABCD$ can be inscribed in a circle with center O. Then $OA = OB = OC = OD$ because all four vertices lie on the circle. Because $ABCD$ is a rhombus, $AB = BC = CD = DA$. Then $\triangle ABC \cong \triangle BCD \cong \triangle CDA \cong \triangle DAB$ by SSS congruence, so $\angle A \cong \angle B \cong \angle C \cong \angle D$. Since the measures of the four angles of a quadrilateral sum to $360°$, each of the four angles is a right angle, and the quadrilateral is a square.



6. In the figure shown, O is the center of the circle, $\overline{AB} \parallel \overline{CD}$, $AB = 24$, $OM = 5$, $CD = 10$, and M is the midpoint of $\overline{AB}$. Find MN.



When I draw in $\overline{OA}$ and $\overline{OD}$, I create right triangles $\triangle AMO$ and $\triangle OND$. I can use the Pythagorean theorem to figure out the length of $\overline{OA}$, which is the same as the length of $\overline{OD}$ since both segments are radii of the circle.



Since M is the midpoint of $\overline{AB}$, $AM = MB = 12$. Then $\triangle AMO$ is a right triangle with legs of lengths 5 and 12. By the Pythagorean theorem, $OA = \sqrt{5^2 + 12^2} = 13$. Since $\overline{OA}$ and $\overline{OD}$ are radii of the same circle, $OD = 13$.

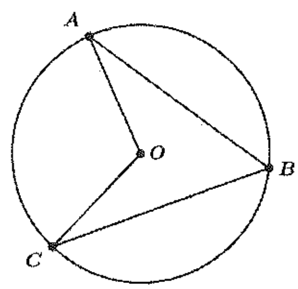
Because the diameter through M bisects chord $\overline{AB}$, $\overline{MN} \perp \overline{AB}$. It follows that N is the midpoint of $\overline{CD}$, so $CN = ND = 5$.

Because $\overline{AB} \parallel \overline{CD}$, and $\overline{MN} \perp \overline{AB}$, $\overline{MN}$ is also perpendicular to $\overline{CD}$; thus, $\angle OND$ is a right angle. Then $\triangle OND$ is a right triangle with a leg of length 5 and hypotenuse of length 13. By the Pythagorean theorem, $13^2 = 5^2 + (ON)^2$, so $ON = 12$.

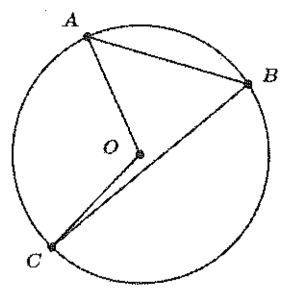
Therefore, $MN = OM + ON = 5 + 12$, so $MN = 17$.

Lesson 4: Experiments with Inscribed Angles

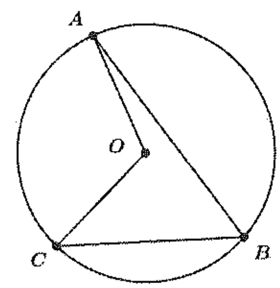
1. Use a protractor to measure the inscribed angle $\angle ABC$ and the central angle $\angle AOC$ for each figure below.



$m\angle AOC = 112°$
$m\angle ABC = 56°$



$m\angle AOC = 112°$
$m\angle ABC = 56°$



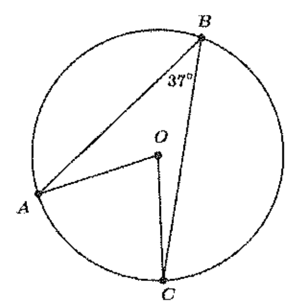
$m\angle AOC = 112°$
$m\angle ABC = 56°$

Since the measures of the central angles are the same in each figure, I expect that the measures of the inscribed angles are also the same based on the results of the exercises done in class.

2. In the figure shown, O is the center of the circle. Find $m\angle AOC$.

$m\angle AOC = 2 \cdot m\angle ABC$
$m\angle AOC = 2(37°)$
$m\angle AOC = 74°$

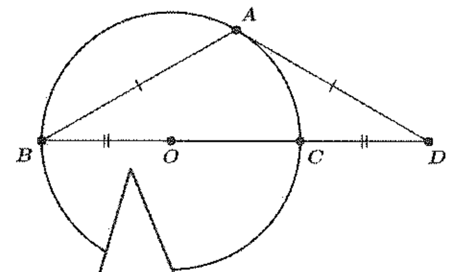
I know the measure of the inscribed angle, and we saw in the lesson that the measure of the central angle is double the measure of the inscribed angle.



3. An inscribed angle cuts off an arc with length $\frac{1}{12}$ of the circumference of the circle. What is the measure of the inscribed angle?

Because the inscribed angle cuts off an arc with length $\frac{1}{12}$ of the circumference of the circle, we know that the measure of the central angle is $\frac{1}{12}$ of $360°$. Thus, the central angle has measure $30°$, and the inscribed angle has measure $15°$.

4. In the figure shown, O is the center of the circle, $\overline{AB} \cong \overline{AD}$, and $\overline{BO} \cong \overline{DC}$. Find $m\angle ABC$.



Because $\overline{AB} \cong \overline{AD}$, $\triangle ABD$ is an isosceles triangle; thus, $\angle ABD \cong \angle ADB$.

Draw $\overline{OA}$ and $\overline{AC}$. By SAS congruence, $\triangle AOB \cong \triangle ACD$, so $\overline{OA} \cong \overline{AC}$.

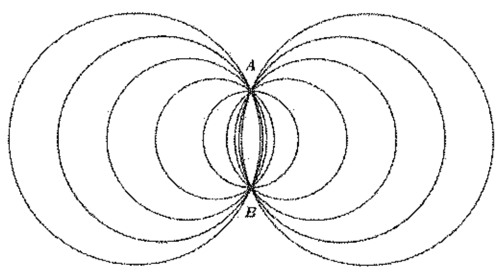
However, $\overline{OA}$ is a radius of the circle, so $\overline{OA} \cong \overline{OC}$. Thus, $\triangle AOC$ is an equilateral triangle, and $m\angle OAC = 60°$.

Since $\angle BAC$ is inscribed in a semicircle, $m\angle BAC = 90°$. Therefore, $m\angle BAO = m\angle BAC - m\angle OAC = 90° - 60° = 30°$.

Since $\overline{OA} \cong \overline{OB}$, $\triangle AOB$ is isosceles, and $\angle ABC \cong \angle BAO$. Thus, $m\angle ABC = 30°$.

> Drawing in $\overline{OA}$ and $\overline{AC}$ gives me more triangles to work with. I can see that $\triangle BAC$ is a right triangle and that $\triangle AOB$ is isosceles, and those tell me something about the angle measures.

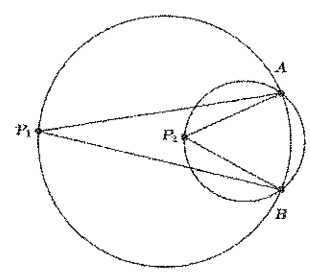
5. There are an infinite number of circles that pass through two given points in the plane.



> I see that the circles can have a radius as big as I want, but that the smallest circle has diameter AB.

Suppose that A and B are points in the plane, and that P is any point on a circle of radius r that passes through A and B. If r increases, does $m\angle APB$ increase or decrease? Explain how you know.

Consider the figure at right, in which P_1 is a point on the circle with larger radius and P_2 is a point on the circle with smaller radius. We can see that $m\angle AP_1B < m\angle AP_2B$, and it appears that as the radius increases, the measure of the inscribed angle decreases.



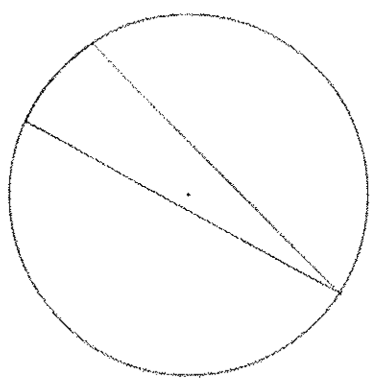
More precisely, as the radius increases, the length of $\widehat{AB}$ becomes a smaller fraction of the total circumference of the circle. In addition, the measure of the central angle is a smaller fraction of $360°$, so the measure of the central angle decreases. Since the measure of the inscribed angle is half of the measure of the central angle, the measure of the inscribed angle also decreases.

Lesson 5: Inscribed Angle Theorem and Its Applications

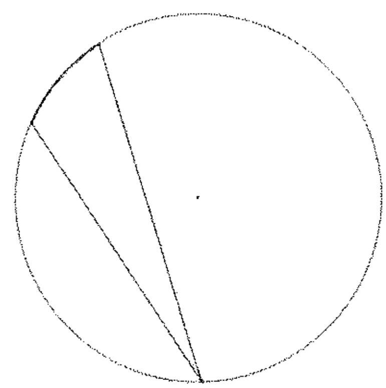
1. An inscribed angle is an angle whose vertex is on a circle, and each side of the angle intersects the circle in another point.

   Draw an example of the different cases an inscribed angle can be positioned. An inscribed angle can be positioned such that

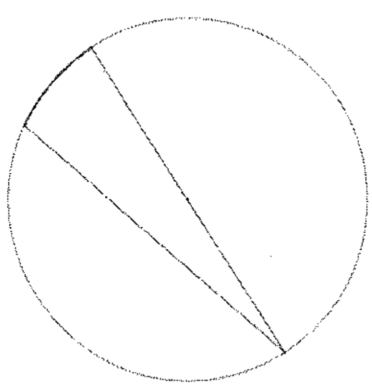
   a. The center of the circle is in the interior of the angle.



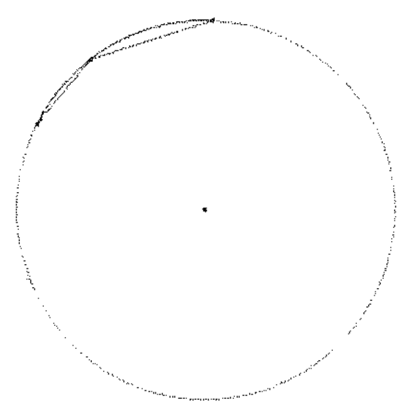
   b. The center of the circle is in the exterior of the angle.



   c. The center is on a side of the angle.

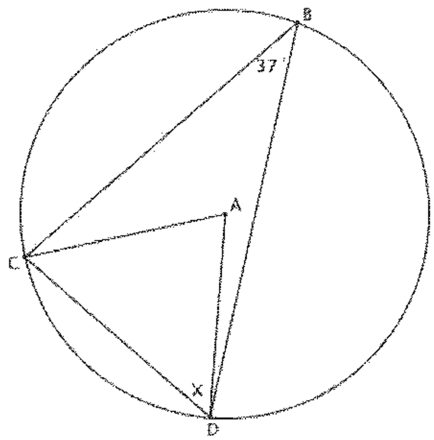


   d. The vertex of the angle is on the minor arc that the inscribed angle intercepts.



Solve for the unknown angle measure in each figure.

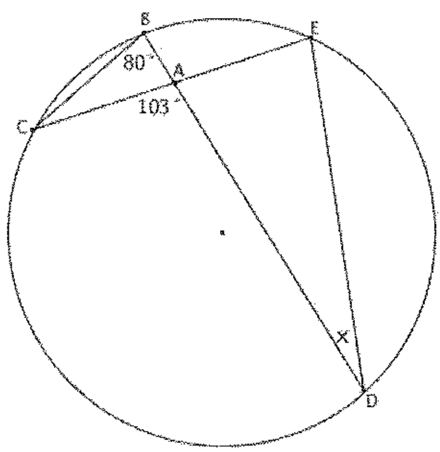
2.



By the inscribed angle theorem, the measure of central angle $\angle CAD$ is two times the measure of inscribed angle $\angle CBD$. Since $\angle CBD$ is $37°$, $\angle CAD$ is $74°$. $\triangle ACD$ is an isosceles triangle with sides $AC = AD$ since both lengths are radii, and base angles $\angle ACD$ and $\angle ADC$ are equal in measure. By the triangle sum theorem, x is calculated to be $53°$.

The inscribed angle theorem states that the measure of a central angle is twice the measure of any inscribed angle that intercepts the same arc as the central angle.

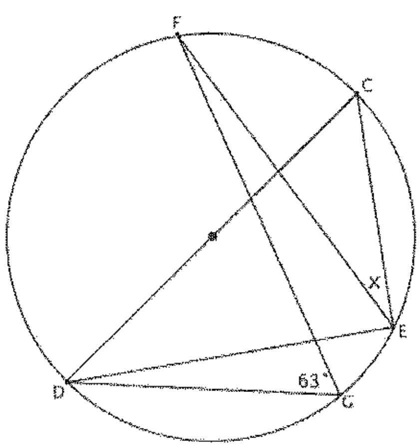
3.



Since $\angle CBD$ and $\angle CED$ both intercept minor arc $\widehat{CD}$, $\angle CED$ must also have a measure of $80°$. Since $\angle EAD$ is supplementary to $\angle CAD$, which has a measure of $103°$, the measure of $\angle EAD$ is $77°$. By the triangle sum theorem, x is calculated to be $23°$.

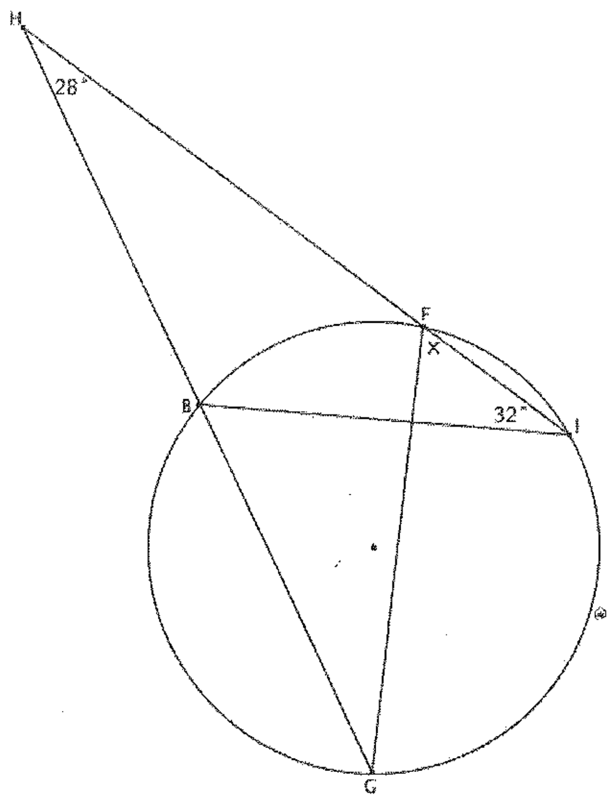
I must use a combination of angle relationships (e.g., inscribed angles, angles in a triangle, angles on a line) in order to solve for x.

4.



Since an angle inscribed in a semicircle is a $90°$ angle, $\angle CED$ must have a measure of $90°$. Inscribed angles $\angle DGF$ and $\angle DEF$ intercept the same arc and, therefore, both have a measure of $63°$. Therefore, x can be calculated to be $27°$.

5.

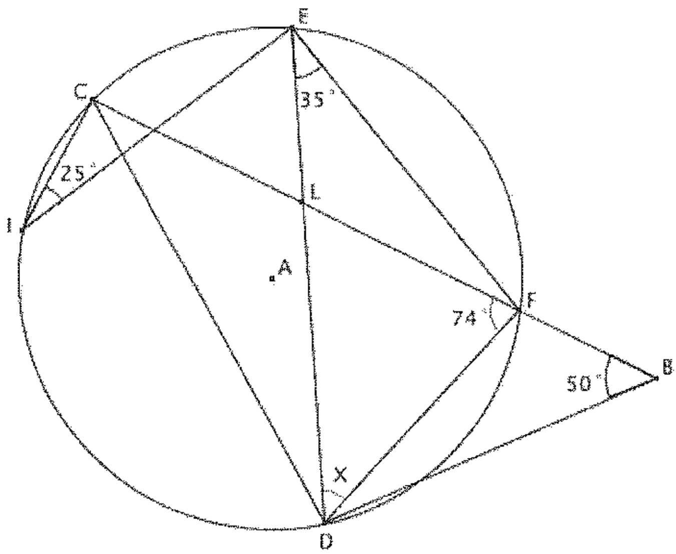


Since inscribed angles $\angle BGF$ and $\angle FIB$ both intercept minor arc $\widehat{BF}$, they both have a measure of $32°$. By the triangle sum theorem, the measure of $\angle HFG$ must be $120°$. Since $\angle HFG$ is a supplement to $\angle IFG$, which is assigned x, $\angle IFG$ must have a measure of $60°$.

Lesson 6: Unknown Angle Problems with Inscribed Angles in Circles

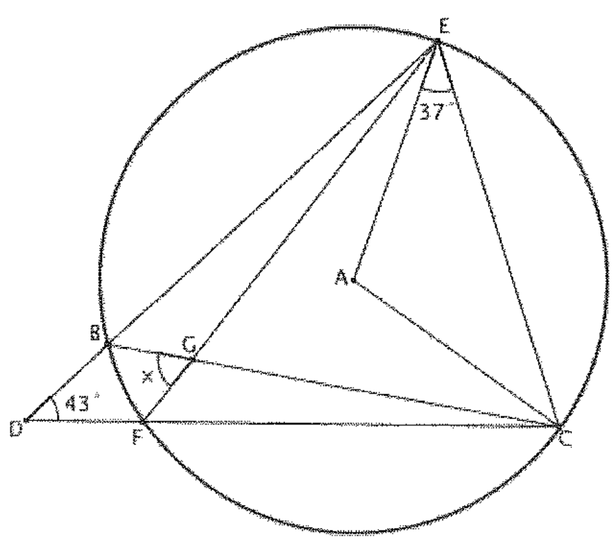
Solve for the unknown angle measure in each figure.

1.



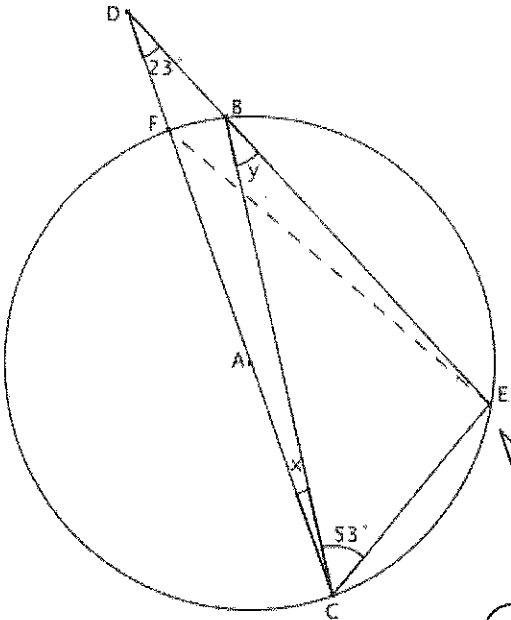
Since $\angle FED$ and $\angle FCD$ both intercept minor arc $\widehat{FD}$, $\angle FCD$ must have an angle measure of $35°$. Since $\angle CIE$ and $\angle CDE$ both intercept minor arc $\widehat{CE}$, $\angle CDE$ must have an angle measure of $25°$. By the triangle sum theorem, x is calculated to be $46°$.

2.



In isosceles $\triangle EAC$, the measure of one base angle is $37°$, which means the measure of $\angle EAC$ is $106°$. Inscribed angles $\angle EBC$ and $\angle CFE$ both intercept the same minor arc as central $\angle EAC$. Since the measure of the central angle must be twice the measure of the inscribed angle that intercepts the same arc, the measures of $\angle EBC$ and $\angle EFC$ are both $53°$. This means the supplements to each of these angles, $\angle DBC$ and $\angle DFE$, both have a measure of $127°$. Since the sum of the angles of a quadrilateral is $360°$, x must be $63°$.

3.



Since $\angle EBC$ and $\angle EFC$ both intercept minor arc $\widehat{EC}$, $\angle EFC$ must have a measure of y. $\angle FEC$ is inscribed in a semicircle, and so the measure of $\angle FEC$ is $90°$. Then, by the triangle sum theorem, $90° + 53° + x + y = 180°$. From $\triangle BDC$, $y = x + 23°$ because the exterior angle of a triangle is equal to the sum of the remote interior angles. Substitute the expression $(x + 23°)$ for y to rewrite the angle measure sum for $\triangle EFC$:

$$90° + 53° + x + x + 23° = 180°$$
$$166° + 2x = 180°$$
$$x = 7°$$

Therefore, $y = 30°$.

I must remember that an angle inscribed in a semicircle has a measure of $90°$, as in $\triangle EFC$.

I must notice that there are two inscribed angles that intercept the same minor arc, $\widehat{EC}$.

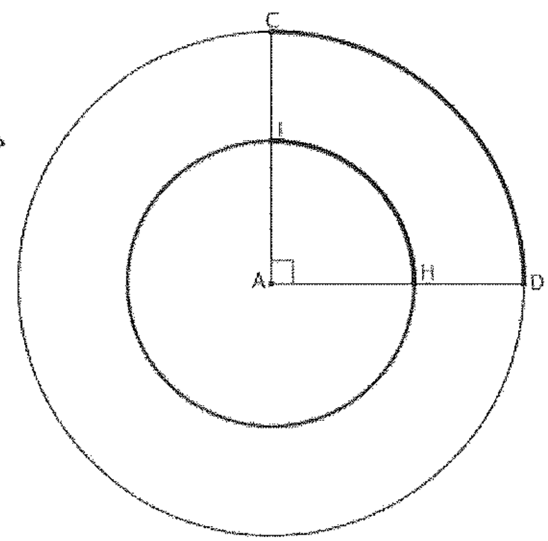
Lesson 7: The Angle Measure of an Arc

1. True or False: Two arcs with the same arc measure must have the same arc length. Include a sketch to justify your answer.

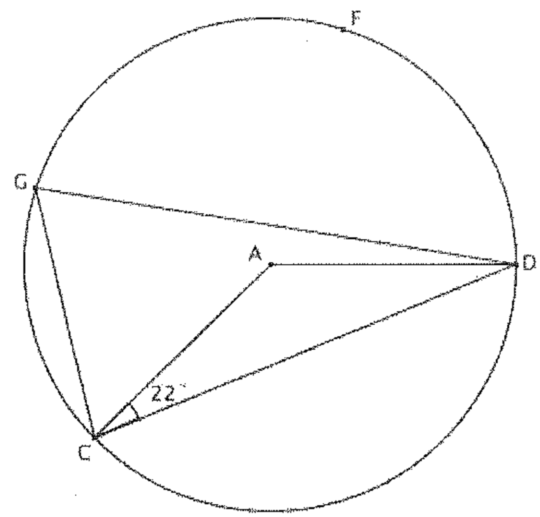
   False. Circles of different sizes (i.e., of different radii length) can have arcs of the same arc measure but of different arc length. This is because the central angle that intercepts the respective arc of each circle can be the same angle for both circles, as in the sketch below.

   In this sketch, central angle $\angle CAD$ has a measure of $90°$ and intercepts $\widehat{IH}$ and $\widehat{CD}$; both arcs have an angle measure of $90°$. $\widehat{IH}$ and $\widehat{CD}$ do not, however, have the same arc length. If we were to lay a string along each arc and then straighten each piece of string, the string laid along $\widehat{CD}$ would be longer than the string laid along $\widehat{IH}$.

The angle measure of an arc is the amount of turning that the arc represents, whereas the arc length can be measured in the same units that we use to measure lengths along straight edges, such as inches and centimeters.



2. In circle A, find the following angle measures.



a. $m\widehat{CD}$

Since $\triangle ACD$ is isosceles, the measures of $\angle ACD$ and $\angle ADC$ are each $22°$. This means $\angle A$ has a measure of $136°$. Then the angle measure of $\widehat{CD}$ is also $136°$.

b. $m\widehat{CFD}$

$224° = 360° - 136°$

The angle measure of $\widehat{CFD}$ is $224°$.

The angle measure of the major arc can be found by subtracting the angle measure of the corresponding minor arc from 360°.

c. $m\angle CGD$

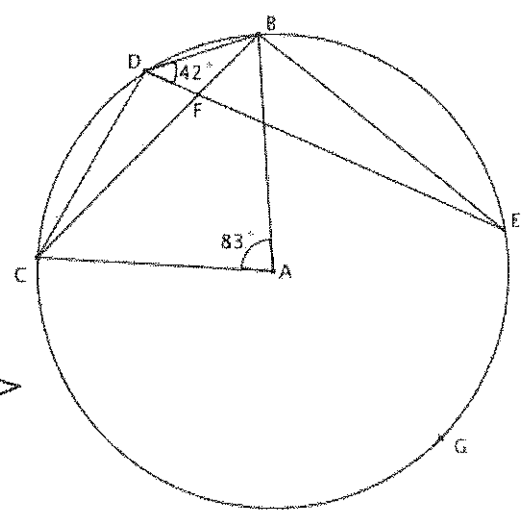
$$m\angle CGD = \frac{1}{2}(m\angle CAD)$$

$$68° = \frac{1}{2}(136°)$$

The measure of the central angle is double the measure of any inscribed angle that intercepts the same arc. The measure of $\angle CGD$ is $68°$.

3. In the figure, $m\angle BAC = 83°$ and $m\angle BDE = 42°$. Find $m\angle CDE$.

Since the measure of central angle $\angle BAC$ is $83°$, the angle measure of $\widehat{BEC}$ is $277°$ because $360° - 83° = 277°$. Inscribed angle $\angle BDE$ has a measure of $42°$, which means that the angle measure of $\widehat{BE}$ is $84°$. The angle measure of $\widehat{CGE}$ is $193°$ because $277° - 84° = 193°$. Then inscribed angle $\angle CDE$ that intercepts $\widehat{CGE}$ must have a measure of $96.5°$.



The angle measure of $\widehat{BE}$ can be subtracted from the angle measure of $\widehat{BEC}$ since they are overlapping arcs. This will aid us in finding the measure of $\angle CDE$.

Lesson 8: Arcs and Chords

1. In the following figure, $\overline{DE} \parallel \overline{FG}$ and the measure of $\widehat{CF}$ is $180°$. Find the angle measure of each arc.

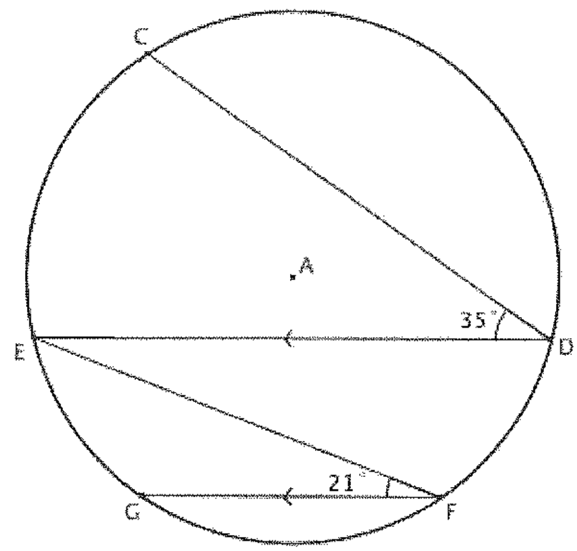
   a. $m\widehat{FG}$

   Since $\overline{DE} \parallel \overline{FG}$, the measures of $\angle DEF$ and $\angle EFG$ are equal, $21°$ each. Since each is an inscribed angle, the measures of intercepted arcs $\widehat{EG}$ and $\widehat{DF}$ are each $42°$. Similarly the measure of $\widehat{CE}$ is $70°$ since inscribed angle $\angle EDC$ has a measure of $35°$. Then the measure of $\widehat{FG}$ is

   $$m\widehat{FG} = m\widehat{CF} - m\widehat{CE} - m\widehat{EG}$$
   $$m\widehat{FG} = 180° - 70° - 42°$$
   $$m\widehat{FG} = 68°$$

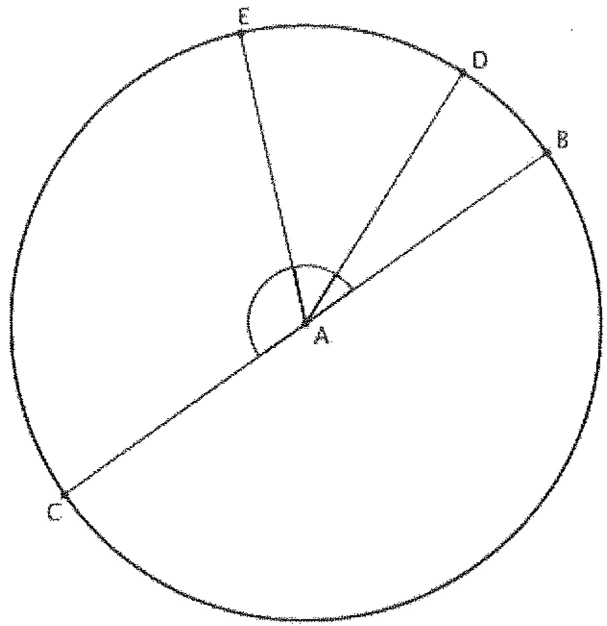
   b. $m\widehat{CD}$

   $$m\widehat{CD} = 360° - m\widehat{CF} - m\widehat{DF}$$
   $$m\widehat{CD} = 360° - 180° - 42°$$
   $$m\widehat{CD} = 138°$$



I must remember that the measures of the arcs between parallel chords are equal.

2. $\overline{BC}$ is the diameter of circle A. $m\widehat{BD}:m\widehat{DE}:m\widehat{EC} = 1:2:5$. Find the angle measure of each arc.



Let x represent the angle measure of $\widehat{BD}$. Then $2x$ and $5x$ represent the measures of $\widehat{DE}$ and $\widehat{EC}$, respectively.

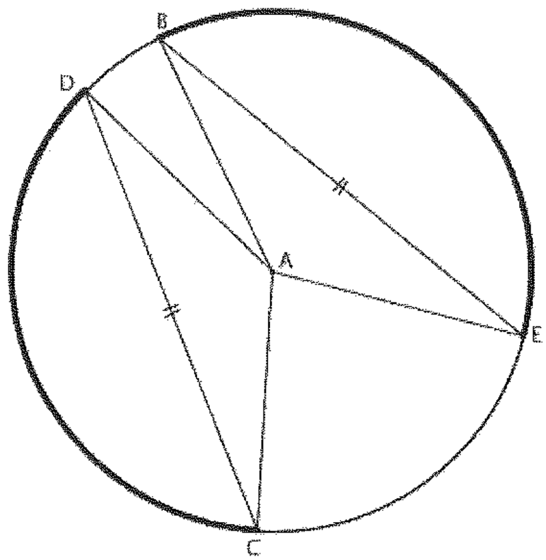
$$x + 2x + 5x = 180^\circ$$
$$8x = 180^\circ$$
$$x = 22.5^\circ$$

$m\widehat{BD} = 22.5^\circ$

$m\widehat{DE} = 2(22.5^\circ) = 45^\circ$

$m\widehat{EC} = 5(22.5^\circ) = 112.5^\circ$

3. In a circle, chords of equal length are said to subtend arcs of equal measure. Use the following figure, in which $DC = BE$, to demonstrate why this must be true.



The arc subtended by a chord includes the endpoints of the chord where it meets the circle and all the points on the circle in between the endpoints.

It is given that $\overline{DC}$ is equal in length to $\overline{BE}$. $\overline{AD}$, $\overline{AC}$, $\overline{AB}$, and $\overline{AE}$ are radii and, therefore, are all equal in length. Then $\triangle ADC \cong \triangle ABE$ by SSS. The measures of $\angle DAC$ and $\angle BAE$ are equal since corresponding angles of congruent triangles are equal in measure. Furthermore, these angles are central angles of equal measure and, therefore, intercept arcs of equal measure. Thus, we have shown that chords of equal length subtend arcs of equal measure.

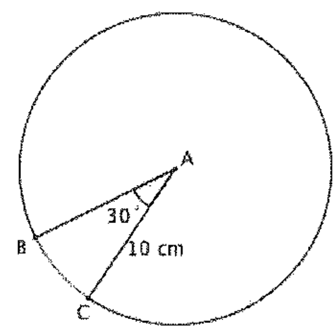
Lesson 9: Arc Length and Areas of Sectors

1. What is the difference between the angle measure of an arc and an arc length?

   The length of an arc is the circular distance around the arc, whereas the angle measure of an arc represents the amount of turning that an arc requires.

2. In circle A, $\angle BAC$ has a measure of $30°$, and radius $\overline{AC}$ has a length of 10 cm.

   a. What is the arc length of $\widehat{BC}$?

   $$\text{Arc length} = \left(\frac{30}{360}\right)(2\pi)(10)$$

   $$\text{Arc length} = \frac{5\pi}{3}$$

   $\widehat{BC}$ has a length of $\frac{5\pi}{3}$ cm.

   b. What is the area of the sector BAC?

   $$\text{Area (sector } BAC) = \left(\frac{30}{360}\right)(\pi)(10)^2$$

   $$\text{Area (sector } BAC) = \frac{25\pi}{3}$$

   Sector BAC has an area of $\frac{25\pi}{3}$ cm^2.

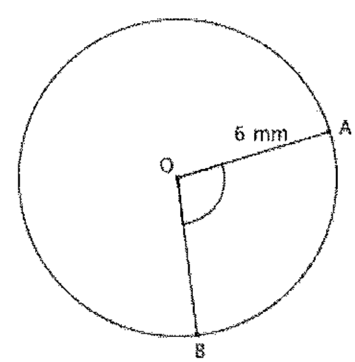
3. Circle O has a radius of length 6 mm.

   What is the measure of the intercepted arc $\widehat{AB}$ that has an arc length of $\frac{10\pi}{3}$ cm?

   $$\frac{10\pi}{3} = \left(\frac{x}{360}\right)(2\pi)(6)$$

   $$x = 100$$

   $\widehat{AB}$ has a measure of $100°$.



4. The following circle has a radius of 5 in., and the angle measure of $\widehat{BC}$ is $60°$. Find the area of the shaded region.

$$\text{Area (sector } BAC) = \left(\frac{60}{360}\right)(\pi)(5)^2$$

$$\text{Area (sector } BAC) = \frac{25\pi}{6}$$

Sector BAC has an area of $\frac{25\pi}{6}$ in^2.

$$\text{Area } (\triangle BAC) = \frac{1}{2}(5)(2.5\sqrt{3})$$

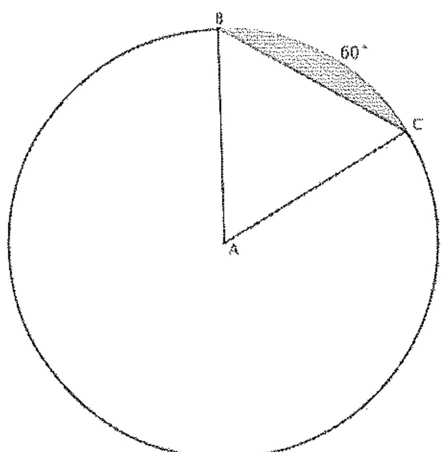
$$\text{Area } (\triangle BAC) = 6.25\sqrt{3}$$

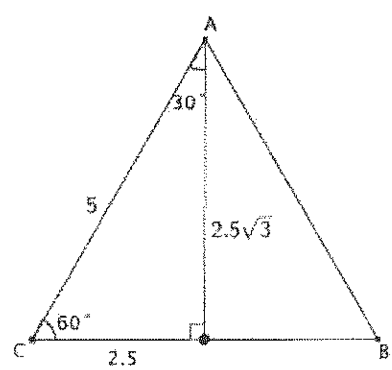
The base and height are made clear by drawing the 30–60–90 triangle within one half of $\triangle BAC$, as shown in the enlarged image below.

$$\text{Area (Shaded Region)} = \frac{25\pi}{6} - 6.25\sqrt{3}$$

$$\text{Area (Shaded Region)} \approx 2.265$$

The area of the shaded region is approximately 2.265 in^2.





The area of the shaded region is equal to the area of the sector minus the area of $\triangle ABC$.

5. Circle O has a radius r and an arc $\widehat{AB}$. Fill in the table below.

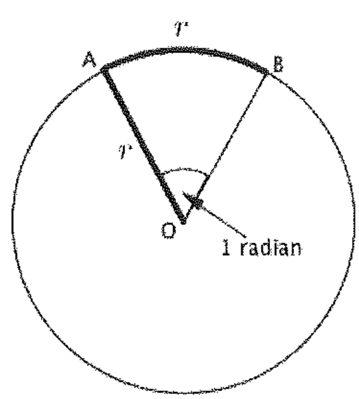
Angle Measure of $\widehat{AB}$	Arc Length of $\widehat{AB}$ Calculation	Arc Length of $\widehat{AB}$
$20°$	$\text{arc length} = \frac{20}{360} \cdot 2\pi r$	$\text{arc length} = \frac{\pi}{9} r$
$36°$	$\text{arc length} = \frac{36}{360} \cdot 2\pi r$	$\text{arc length} = \frac{\pi}{5} r$
$60°$	$\text{arc length} = \frac{60}{360} \cdot 2\pi r$	$\text{arc length} = \frac{\pi}{3} r$
$144°$	$\text{arc length} = \frac{144}{360} \cdot 2\pi r$	$\text{arc length} = \frac{4\pi}{5} r$
$x°$	$\text{arc length} = \frac{x}{360} \cdot 2\pi r$	$\text{arc length} = \frac{\pi x}{180} r$

Given an angle measure, to what is arc length proportional? What is the constant of proportionality in the calculation of the arc length?

Given an angle measure, arc length is proportional to the length of the radius. Since all circles are similar, a central angle of $1°$ subtends an arc of length $\frac{\pi}{180}$ multiplied by the radius. The constant of proportionality is $\frac{\pi}{180}$.

6. What is a radian?

Radians are a type of angle measure, just as degrees are a type of angle measure. One radian is the measure of the central angle of a sector of a circle with arc length equal to one radius. In the figure below, the measure of $\angle AOB$ is 1 radian.



I can imagine laying a string along the radius and along $\widehat{AB}$. If I were to compare the two pieces of string, they would be equal in length.

Lesson 10: Unknown Length and Area Problems

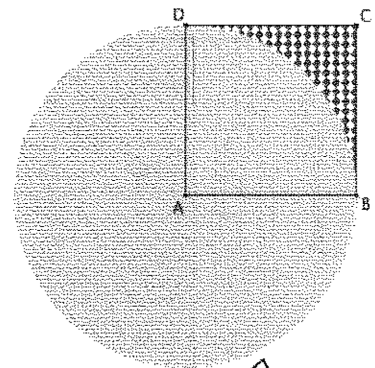
1. Circle A has a radius of length 12 mm. Center A is one vertex of square $ABCD$, and points B and D lie on circle A.

   a. What is the arc length of $\widehat{BD}$?

   $\text{Arc length}(\widehat{BD}) = \left(\frac{90}{360}\right)(2\pi)(12)$

   $\text{Arc length}(\widehat{BD}) = 6\pi$

   The length of $\widehat{BD}$ *is* 6π mm.



The dotted region lies inside the square but outside the quarter-circle that lies within the square.

   b. What is the area of the dotted region defined by B, C, and D.

   $\text{Area}(ABCD) = 12^2$

   $\text{Area}(ABCD) = 144$

   $\text{Area}(\text{sector } DAB) = \left(\frac{90}{360}\right)(\pi)(12)^2$

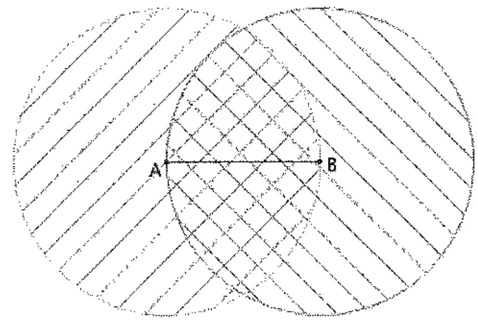
   $\text{Area (sector } DAB) = 36\pi$

   $\text{Area (dotted region)} = 144 - 36\pi$

   $\text{Area (dotted region)} \approx 30.9$

   The dotted region has an approximate area of $30.9\ \text{mm}^2$.

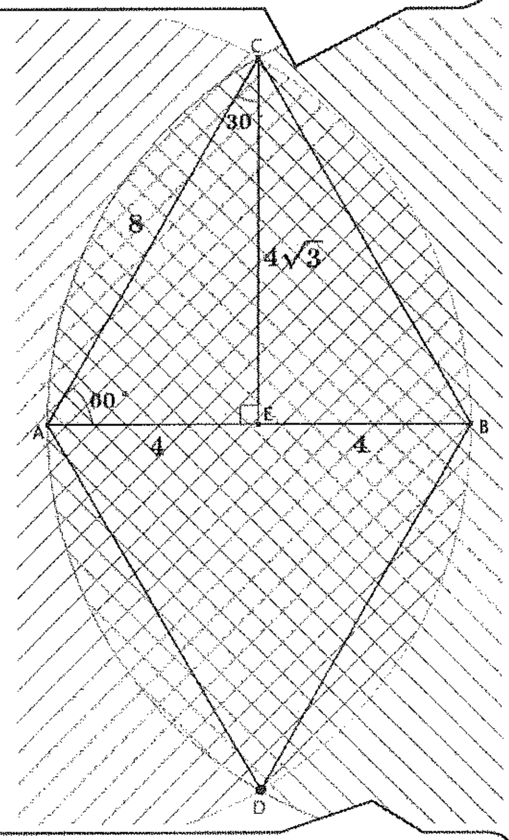
2. Circles A and B each have a radius of 8 units. What is the area of the double-shaded region?



$$\text{Area}(\triangle ABC) = \left(\frac{1}{2}\right)(8)(4\sqrt{3})$$

$$\text{Area}(\triangle ABC) = 16\sqrt{3}$$

I can draw triangles ABC and ABD after locating C and D using the steps for the construction of an equilateral triangle. Then I can find the height of each triangle by drawing a 30–60–90 triangle in one half of $\triangle ABC$, as shown below.



Triangles ABC and ABD are congruent by SSS.

Area of both triangles:

$$\text{Area}(\triangle ABC) + \text{Area}(\triangle ABD) = 2\left(16\sqrt{3}\right)$$

$$\text{Area}(\triangle ABC) + \text{Area}(\triangle ABD) = 32\sqrt{3}$$

Area of the sliver, A_s, defined by $\overline{AC}$ and $\widehat{AC}$:

$$\text{Area}(A_s) = \text{Area}(\text{sector } ABC) - \text{Area}(\triangle ABC)$$

$$\text{Area}(A_s) = \left(\frac{60}{360}\right)(\pi)(8)^2 - 16\sqrt{3}$$

$$\text{Area}(A_s) = \frac{32\pi}{3} - 16\sqrt{3}$$

There are four such slivers in the double-shaded region.

Total area, A, of these four regions:

$$\text{Area}(A) = 4\left(\frac{32\pi}{3} - 16\sqrt{3}\right)$$

The total area of the double-shaded region is the area of the two triangles plus the area of the four slivers.

$$\text{Area(shaded)} = 32\sqrt{3} + 4\left(\frac{32\pi}{3} - 16\sqrt{3}\right)$$

$$\text{Area(shaded)} = 32\left(\sqrt{3} + \frac{4\pi}{3} - 2\sqrt{3}\right)$$

$$\text{Area(shaded)} = 32\left(\frac{4\pi}{3} - \sqrt{3}\right)$$

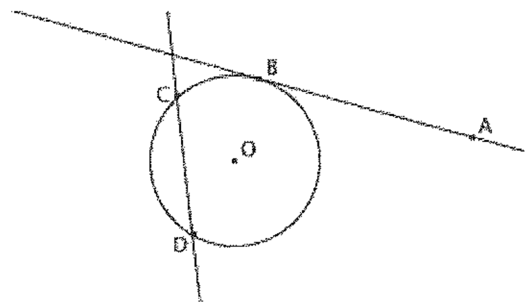
The area of the shaded region is $32\left(\frac{4\pi}{3} - \sqrt{3}\right)$ units^2, *which is approximately* 78.6 units^2.

I need to decompose the double-shaded region into regions where the area can be found. I can do this by using a combination of equilateral triangles and sectors.

Lesson 11: Properties of Tangents

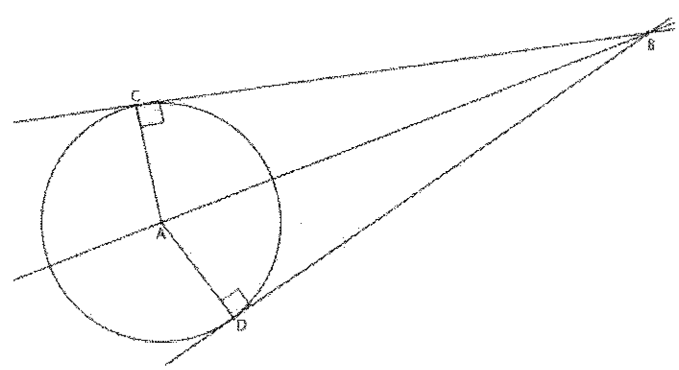
1. For circle O, sketch a line, $\overleftrightarrow{AB}$, tangent to the circle at point B. Sketch a secant line, $\overleftrightarrow{CD}$, intersecting the circle at points C and D. What distinguishes a tangent line from a secant line?

   Possible sketch:



   A tangent line to a circle is a line in the same plane that intersects the circle in one and only one point. A secant line to a circle is a line in the same plane that intersects the circle in exactly two points.

2. In the following figure, $\overleftrightarrow{BC}$ and $\overleftrightarrow{BD}$ are tangent to circle A at points C and D, respectively. If the radius length of the circle is 7 units, and the length of $\overline{AB}$ is 25 units, find:



   a. BC.

   It is given that $\overleftrightarrow{BC}$ and $\overleftrightarrow{BD}$ are tangent to circle A, then $\overleftrightarrow{BC}$ is perpendicular to $\overline{AC}$ and $\overleftrightarrow{BD}$ is perpendicular to $\overline{AD}$, making $\triangle ABC$ and $\triangle ABD$ each right triangles. So the side lengths should satisfy the Pythagorean theorem:

$$25^2 = 7^2 + (BC)^2$$
$$BC = 24$$

   The length of $\overline{BC}$ is 24 units.

I must remember that a tangent line to a circle is perpendicular to the radius of the circle drawn to the point of tangency.

b. BD.

Since $\triangle ABC$ and $\triangle ABD$ are congruent right triangles by HL, the length of $\overline{BD}$ is 24 units.

c. Perimeter of $ACBD$.

$\text{Perimeter}(ACBD) = AC + CB + BD + DA$

$\text{Perimeter}(ACBD) = 7 + 24 + 24 + 7$

$\text{Perimeter}(ACBD) = 62$

The perimeter of $ACBD$ is 62 units.

d. Area of $ACBD$.

$\text{Area}(ACBD) = \text{Area}(\triangle ABC) + \text{Area}(\triangle ABD)$

$\text{Area}(\triangle ABC) = \frac{1}{2}(7)(24)$

$\text{Area}(\triangle ABC) = 84$

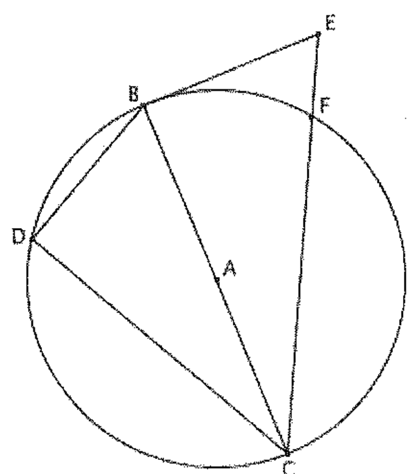
Since $\triangle ABC$ and $\triangle ABD$ are congruent, their areas are equal.

$\text{Area}(ACBD) = 84 + 84$

$\text{Area}(ACBD) = 168$

The area of $ACBD$ is 168 square units.

3. In circle A, $\overline{BE}$ is tangent to the circle, and $\overline{BC}$ is a diameter. Point B is the midpoint of $\widehat{DF}$. Triangles BCD and ECB are similar. Explain why.



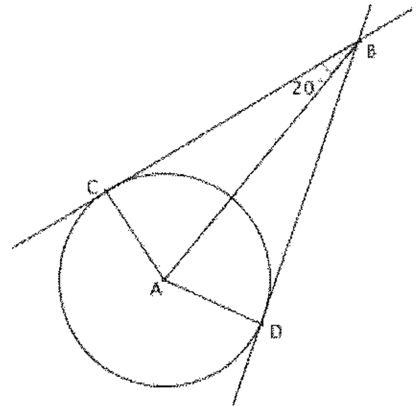
Since inscribed angle $\angle BDC$ is inscribed within a semicircle, $\angle D$ must be a right angle. Since $\overline{BE}$ is tangent to circle A, $\overline{BE}$ is perpendicular to diameter $\overline{AC}$; therefore, $\angle EBC$ is a right angle. It is given that point B is the midpoint of $\widehat{DF}$; therefore, $m\widehat{BD} = m\widehat{BF}$, and furthermore, $m\angle DCB = m\angle BCE$. Then, by the AA similarity criterion, triangles BCD and ECB are similar.

I must remember that inscribed angles that intercept arcs of equal measure are equal in measure.

Lesson 12: Tangent Segments

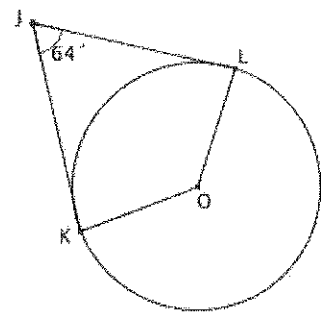
1. The rays of $\angle DBC$ are tangent to circle A. What is the measure of $\angle ABD$? How do you know?

   The measure of $\angle ABD$ is $20°$. If a circle is tangent to both rays of an angle, then its center lies on the angle bisector, which means $\overline{AB}$ is the angle bisector; therefore, $m\angle ABC = m\angle ABD$.



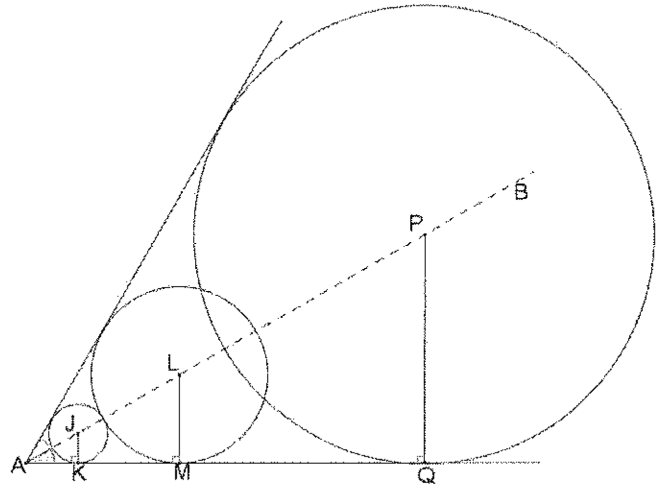
2. $\overline{JL}$ and $\overline{JK}$ are tangent segments to circle O. What is the measure of $\angle JOL$? How do you know?

   $\overline{JO}$ divides $JKOL$ into two congruent right triangles; $\overline{JO}$ is the angle bisector of $\angle KJL$. Then the measure of $\angle LJO$ is $32°$. Since $\overline{JL}$ is tangent to circle O at L, $\angle JLO$ is a right angle. Then, by the triangle sum theorem, the measure of $\angle JOL$ is $58°$.



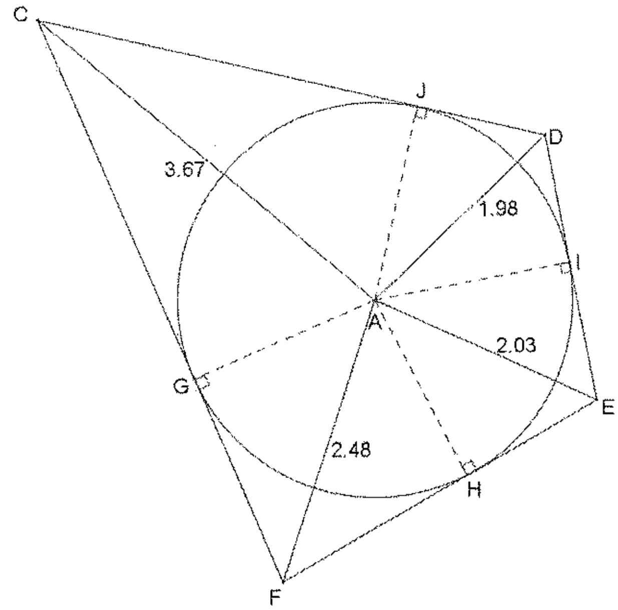
3. An angle, $\angle A$, is provided to the right. Use a compass and straightedge to construct three different circles, each of which is tangent to the rays of $\angle A$. Draw a radius to a point of tangency for each circle. Explain how the construction is performed.

   Construct the angle bisector of $\angle A$. Select a point J on the angle bisector to be the center of a circle. Adjust the compass to the length between J and a ray of $\angle A$ and draw a circle; label the point of intersection with the ray as point K. Draw radius $\overline{JK}$. Repeat these steps with centers in locations other than J.



I should note that the measure of the angle is irrelevant. I need to construct the angle bisector since a circle that is tangent to both rays of a circle must have its center on the angle bisector of the angle.

4. Circle A has a radius of 1.65 and is tangent to quadrilateral $CDEF$ at points J, I, H, and G. Center A is joined to each vertex so that $AC = 3.67$, $AD = 1.98$, $AE = 2.03$, and $AF = 2.48$. What is the perimeter of $CDEF$? Round to the hundredths place.



The following are pairs of congruent, right triangles: $\triangle ACJ$ and $\triangle ACG$, $\triangle ADJ$ and $\triangle ADI$, $\triangle AEI$ and $\triangle AEH$, and $\triangle AFH$ and $\triangle AFG$. Since corresponding parts of congruent triangles are equal in measure, then $CG = CJ$, $DJ = DI$, $IE = HE$, and $HF = GF$. The lengths of these triangles must satisfy the Pythagorean theorem. One length in each of these triangles is the radius length; therefore, only one side length is unknown.

$(1.65)^2 + (CG)^2 = (3.67)^2$; $CG \approx 3.28$

$CJ \approx 3.28$

$(1.65)^2 + (DJ)^2 = (1.98)^2$; $DJ \approx 1.09$

$DI \approx 1.09$

$(1.65)^2 + (IE)^2 = (2.03)^2$; $IE \approx 1.18$

$HE \approx 1.18$

$(1.65)^2 + (HF)^2 = (2.48)^2$; $HF \approx 1.85$

$GF \approx 1.85$

$\text{Perimeter}(CDEF) \approx 2(3.28) + 2(1.09) + 2(1.18) + 2(1.85)$

$\text{Perimeter}(CDEF) \approx 14.8$

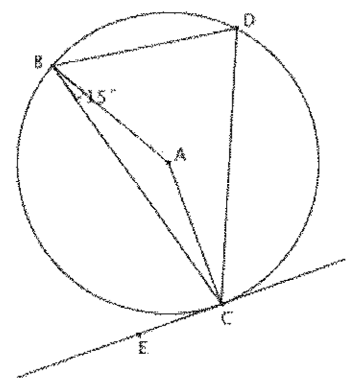
The perimeter of $CDEF$ is approximately 14.8.

The radii drawn to the points of tangency help to outline pairs of congruent right triangles. I can use this information in addition to the radius length to help solve for unknown segment lengths.

Lesson 13: The Inscribed Angle Alternate—A Tangent Angle

1. In circle A, $m\angle ABC = 15°$. Find the measures of $\angle BAC$, $\angle BDC$, and $\angle BCE$.

By the tangent-secant theorem, $m\angle BCE = \frac{1}{2}(m\widehat{BC})$. I can find the measure of the central angle that subtends $\widehat{BC}$ since $\triangle BAC$ must be an isosceles triangle.



Since the measure of $\angle ABC$ is $15°$, and $\angle ABC$ is one base angle of isosceles $\triangle ABC$, the measure of $\angle BAC$ is $150°$, and consequently, $m\widehat{BC} = 150°$.

Inscribed angle $\angle BDC$ intercepts $\widehat{BC}$; therefore, the measure of $\angle BDC$ is $75°$.

By the tangent-secant theorem, the measure of $\angle BCE$ is also $75°$.

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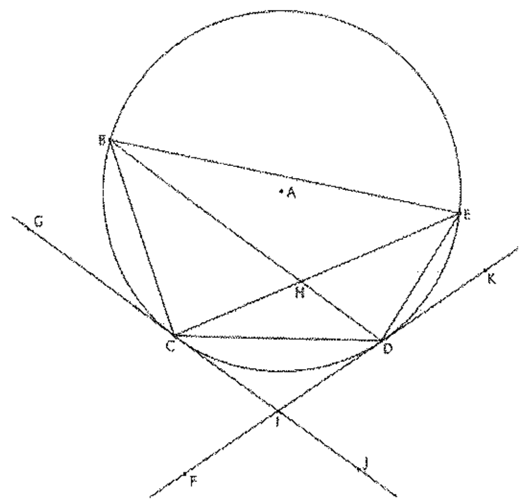
2. $\overleftrightarrow{CG}$ and $\overleftrightarrow{DF}$ are tangent to circle A at C and D. The measure of $\widehat{CD}$ is $70°$, the measure of $\widehat{BE}$ is $170°$, and the measure of $\angle BCG$ is $40°$. Find the measure of each angle and each minor arc of the figure.

I need to consider all types of angle and arc relationships, such as the triangle sum theorem, angles on a line, vertical angles, tangent-secant theorem, and adjacent arcs, to name a few.



Since $m\widehat{CD}$ is $70°$, the inscribed angles that intercept $\widehat{CD}$ are half of $70°$; the measures of $\angle CBD$ and $\angle CED$ are each $35°$. Also, by the tangent-secant theorem, the measures of $\angle DCI$ and $\angle CDI$ are each $35°$. By the triangle sum theorem, $m\angle CID$ is $110°$. It follows that the measure of vertical angle $\angle FIJ$ is $110°$, and the measures of $\angle CIF$ and $\angle DIJ$ are each $70°$.

Since $m\angle BCG$ is $40°$, the measure of $\widehat{BC}$ is $80°$. The measures of $\angle BEC$ and $\angle BDC$ are each $40°$.

Since the measure of $\widehat{BE}$ is $170°$, the measures of $\angle BDE$ and $\angle BCE$ are each $85°$.

With known arc measures for $\widehat{CD}$, $\widehat{BC}$, and $\widehat{BE}$ as $70°$, $80°$, and $170°$, respectively, then $m\widehat{ED}$ must be $40°$. Then the measures of $\angle EBD$, $\angle ECD$, and $\angle EDK$ are each $20°$.

Based on the measures of $\angle ECD$ and $\angle BDC$, $20°$ and $40°$, respectively, the measures of $\angle CHD$ and $\angle BHE$ are each $120°$.

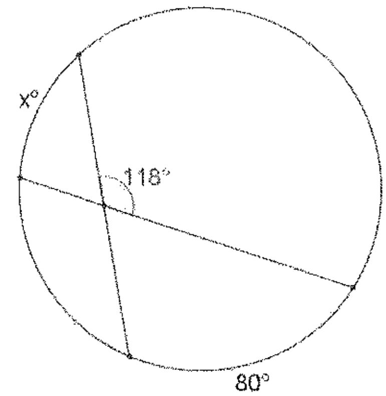
Because $m\widehat{BD} = m\widehat{BC} + m\widehat{CD}$, $\widehat{BD}$ has a measure of $150°$. This means the measure of $\angle BED$ is $75°$. Similarly, because $m\widehat{CE} = m\widehat{CD} + m\widehat{DE}$, $\widehat{CE}$ has an arc measure of $110°$. This means the measure of $\angle EBC$ is $55°$.

$m\widehat{BED} = m\widehat{BE} + m\widehat{ED}$, then $\widehat{BED}$ has a measure of $210°$. This means the measures of $\angle BDK$ and $\angle BCD$ are each $105°$.

Lesson 14: Secant Lines; Secant Lines That Meet Inside a Circle

1. Find the value of x using the information given in the diagram below.

I know that when an angle is formed by two secants in the interior of a circle, the measure of that angle is equal to the average of the measures of the arcs intercepted by the angle and its vertical angle. I need to find the measure of one of those vertical angles first since I only know the measure of one arc.



Let the measure of the angle supplementary to the $118°$ angle be $a°$.

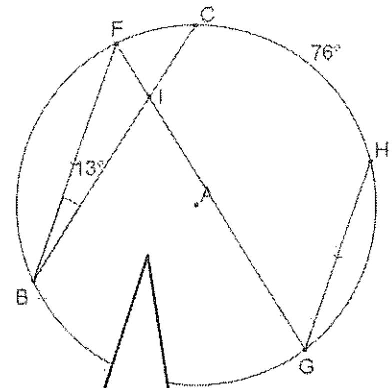
$$118 + a = 180$$
$$a = 62$$

The measure of the angle supplementary to the $118°$ angle is $62°$.

$$62 = \frac{1}{2}(80 + x)$$
$$124 = 80 + x$$
$$x = 44$$

The value of x is 44, so the corresponding arc measure is $44°$.

2. Given the circle with center A, $\overline{BF} \parallel \overline{GH}$, $\overline{GF}$ and $\overline{BC}$ intersect at I, $m\angle FBC = 13°$, and $m\widehat{CH} = 76°$, find $m\angle BIG$.



$m\widehat{FC} = 2(m\angle FBC)$

$m\widehat{FC} = 2(13°)$

$m\widehat{FC} = 26°$ *The measure of arc FC is $26°$.*

$m\widehat{FH} = m\widehat{FC} + m\widehat{CH}$

$m\widehat{FH} = 26° + 76°$

$m\widehat{FH} = 102°$ *The measure of arc FH is $102°$.*

$m\widehat{BG} = m\widehat{FH}$ *Arcs cut by parallel chords have equal measures.*

$m\widehat{BG} = 102°$ *The measure of arc BG is also $102°$.*

$m\angle BIG = \frac{1}{2}\left(m\widehat{BG} + m\widehat{FC}\right)$ *Secant angle theorem, interior case.*

$m\angle BIG = \frac{1}{2}(102° + 26°)$

$m\angle BIG = \frac{1}{2}(128°)$

$m\angle BIG = 64°$

The measure of angle BIG is $64°$.

Angle BIG is formed by two secants that intersect in the circle's interior. To find the measure of the angle, I need to find the measures of arcs BG and FC.

3. $\overline{AB}$ intersects $\overline{CE}$ at point D in the interior of the circle on points A, E, B, and C. Find the measure of angle CDA.

$m\angle CDA = \frac{1}{2}\left(m\widehat{CA} + m\widehat{BE}\right)$

$3x + 1.5 = \frac{1}{2}(4x + 3 + 38)$

$3x + 1.5 = \frac{1}{2}(4x + 41)$

$3x + 1.5 = 2x + 20.5$

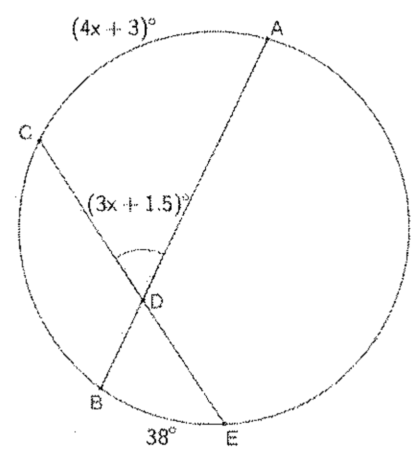
$x = 19$

$m\angle CDA = 3x + 1.5$

$m\angle CDA = 3(19) + 1.5$

$m\angle CDA = 58\frac{1}{2}$

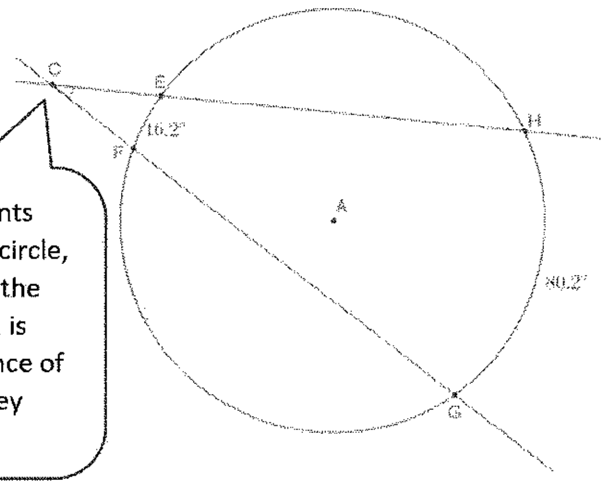
I can use the relationship of the arc measures and the intercepted arcs in the secant angle theorem (interior case) to find the value for x. Then I can substitute the value of x into the expression $(3x + 1.5)$ to find the measure of angle CDA.



The measure of angle CDA is $58\frac{1}{2}°$.

Lesson 15: Secant Angle Theorem, Exterior Case

1. Find $m\angle ECF$.



$$m\angle ECF = \frac{1}{2}\left(m\widehat{GH} - m\widehat{EF}\right)$$

$$m\angle ECF = \frac{1}{2}(80.2° - 16.2°)$$

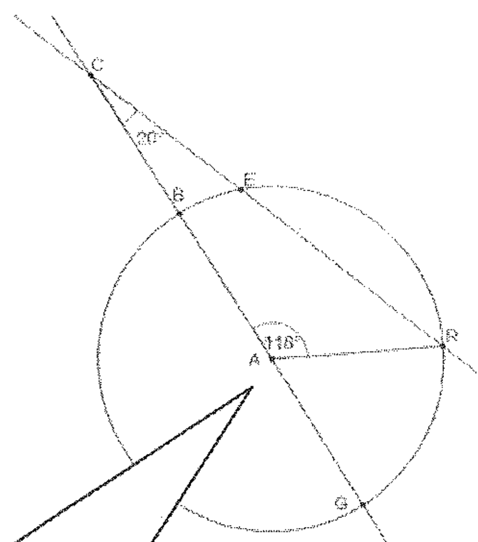
$$m\angle ECF = \frac{1}{2}(64°)$$

$$m\angle ECF = 32°$$

The measure of angle ECF ***is*** $32°$.

When two secants meet outside a circle, the measure of the angle they form is half the difference of the arcs that they intersect.

2. Given the diagram to the right, $\overline{BG}$ is a diameter, and $\overline{AR}$ is a radius. Find $m\widehat{BE}$ and $m\widehat{RG}$.



$\overline{AR}$ is a radius, and $\overline{AB}$ is also a radius; thus, $\angle RAB$ is a central angle, and $m\widehat{BR} = 118°$.

$\overline{BG}$ is a diameter, so $\widehat{BRG}$ is a semicircle and has a measure of $180°$.

Arc measures add, so $m\widehat{BR} + m\widehat{RG} = m\widehat{BRG}$.

$$118° + m\widehat{RG} = 180°$$

$$118° - 118° + m\widehat{RG} = 180° - 118°$$

$$m\widehat{RG} = 62°$$

The measure of $\widehat{RG}$ is $62°$.

$$m\angle GCR = \frac{1}{2}\left(m\widehat{RG} - m\widehat{BE}\right)$$

$$20° = \frac{1}{2}\left(62° - m\widehat{BE}\right)$$

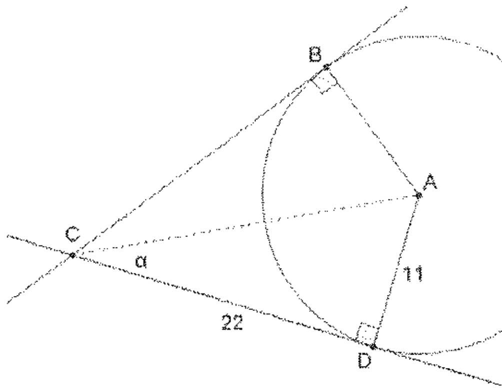
$$40° = 62° - m\widehat{BE}$$

$$m\widehat{BE} = 22°$$

The measure of $\widehat{BE}$ is $22°$.

I know that A is the center of the circle since $\overline{AR}$ is a radius, so $\angle RAB$ is a central angle, and an arc of a circle has the same degree measure as its central angle. I can use this and the arc measure of a semicircle to find the measure of $\widehat{RG}$.

3. Given circle with center A and radius $AD = 11$, and tangents $\overleftrightarrow{CD}$ and $\overleftrightarrow{CB}$ where $CD = 22$, draw the diagram described and find $m\widehat{BD}$.



I know that a diagram will help me understand how all of this information fits together. I should start by drawing a circle with center A and a radius $\overline{AD}$. If D is a point of tangency, then point C must lie somewhere along a line that is perpendicular to $\overline{AD}$ at point D.

Tangents $\overleftrightarrow{CD}$ and $\overleftrightarrow{CB}$ are perpendicular to radii $\overline{AD}$ and $\overline{AB}$, respectively, because tangents to a circle are perpendicular to radii at the points of tangency. It is also true that $CB = CD$ since $\overline{CB}$ and $\overline{CD}$ are tangents to the circle from the same exterior point.

$\overline{AC}$ is then the hypotenuse of right triangles ADC and ABC. Let $m\angle ACD = \alpha$.

$$\tan \alpha = \frac{11}{22}$$
$$\alpha = \arctan \frac{11}{22}$$
$$\alpha \approx 26.6^\circ$$

The measure of α is approximately 26.6°.

$$m\angle BCD = 2\left(\arctan \frac{11}{22}\right)$$

Using the secant angle theorem:

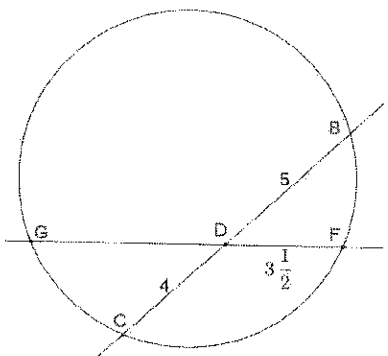
$$m\angle BCD = \frac{1}{2}\left(m\widehat{BED} - m\widehat{BD}\right)$$
$$2\left(\arctan \frac{11}{22}\right) = \frac{1}{2}\left(m\widehat{BED} - m\widehat{BD}\right)$$
$$2\left(\arctan \frac{11}{22}\right) = \frac{1}{2}\left(\left(360^\circ - m\widehat{BD}\right) - m\widehat{BD}\right)$$
$$2\left(\arctan \frac{11}{22}\right) = \frac{1}{2}\left(360^\circ - 2 \cdot m\widehat{BD}\right)$$
$$2\left(\arctan \frac{11}{22}\right) = 180^\circ - m\widehat{BD}$$
$$m\widehat{BD} = 180^\circ - 2\left(\arctan \frac{11}{22}\right)$$
$$m\widehat{BD} \approx 126.8^\circ$$

To find the measure of the arc, I need to know either the measure of the major arc of the circle $\widehat{BED}$ or know the measure of $\angle BCD$. I don't know the arc measure. I do know that $\overline{AD} \perp \overline{CD}$ and $\overline{AB} \perp \overline{CB}$, so I can find $m\angle ACD$ and $m\angle ACB$ using right triangle trigonometry.

The measure of $\widehat{BD}$ is approximately 126.8°.

Lesson 16: Similar Triangles in Circle-Secant (or Circle-Secant-Tangent) Diagrams

1. Given the circle on points B, F, C, and G and secants $\overleftrightarrow{BC}$ and $\overleftrightarrow{GF}$ intersecting at point D in the interior of the circle, if $DB = 5$, $DF = 3\frac{1}{2}$, and $DC = 4$, find DG.



I know that when two secants intersect inside a circle, the lengths of the segments have a special relationship that we found by drawing chords $\overline{GC}$ and $\overline{BF}$ and using AA criterion to prove similar triangles.

Two secant lines intersect inside the circle at D, so the following equation must be true:

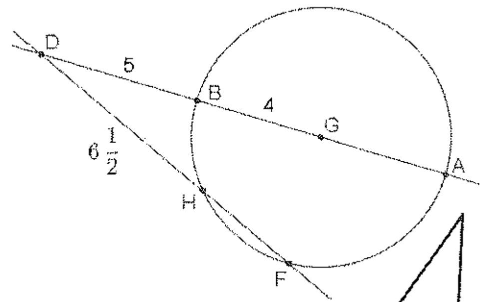
$$DG \cdot DF = DC \cdot DB$$
$$DG \cdot 3\frac{1}{2} = 4 \cdot 5$$
$$DG \cdot 3\frac{1}{2} = 20$$
$$DG \cdot \frac{7}{2} = 20$$
$$DG \cdot \frac{7}{2} \cdot \frac{2}{7} = 20 \cdot \frac{2}{7}$$
$$DG = \frac{40}{7}$$
$$DG = 5\frac{5}{7}$$

The length of $\overline{DG}$ is $5\frac{5}{7}$.

2. Given the circle with center G and radius $BG = 4$, secants $\overleftrightarrow{DBGA}$ and $\overleftrightarrow{DHF}$ intersecting at point D in the exterior of the circle, $DB = 5$, and $DH = 6\frac{1}{2}$, find HF.

$$\boldsymbol{DB \cdot DA = DH \cdot DF}$$
$$\boldsymbol{5 \cdot 13 = 6\frac{1}{2} \cdot DF}$$
$$\boldsymbol{65 = \frac{13}{2} \cdot DF}$$
$$\boldsymbol{\frac{2}{13} \cdot 65 = \frac{2}{13} \cdot \frac{13}{2} \cdot DF}$$
$$\boldsymbol{10 = DF}$$

Secants intersecting outside a circle have segments whose lengths also have a special relationship that we found using similar triangles.



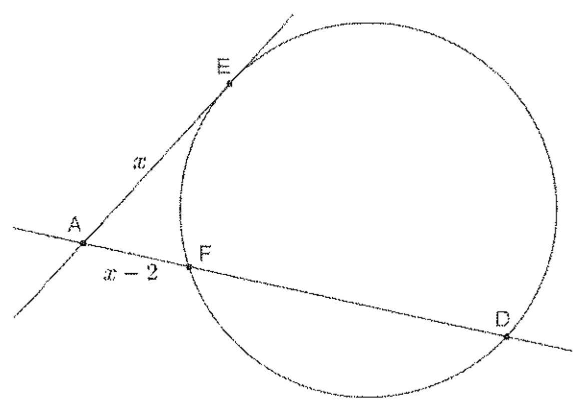
If $\overline{BG}$ is a radius and $\overleftrightarrow{DBGA}$ is a secant, then $\overline{BA}$ is a diameter with length of 8.

$$\boldsymbol{DH + HF = DF}$$
$$\boldsymbol{6\frac{1}{2} + HF = 10}$$
$$\boldsymbol{HF = 3\frac{1}{2}}$$

The length of $\overline{HF}$ is $3\frac{1}{2}$.

3. Given the circle on points E, F, and D, secant $\overleftrightarrow{AFD}$ and tangent $\overleftrightarrow{AE}$, if $AE = x$, $AF = x - 2$, and $AD = 2x$, find AE, AF, FD, and AD.

$$\boldsymbol{AE^2 = AF \cdot AD}$$
$$\boldsymbol{x^2 = (x-2)(2x)}$$
$$\boldsymbol{x^2 = 2x^2 - 4x}$$
$$\boldsymbol{x^2 - 4x = 0}$$
$$\boldsymbol{x(x-4) = 0}$$
$$\boldsymbol{x = 0 \text{ or } x = 4}$$



If $x = 0$, then A, E, F, and D must be the same point.

If $x = 4$, then by substitution, $AE = 4$.

$$\boldsymbol{AF = x - 2}$$
$$\boldsymbol{AF = 4 - 2}$$
$$\boldsymbol{AF = 2}$$

$$\boldsymbol{AD = 2x}$$
$$\boldsymbol{AD = 2(4)}$$
$$\boldsymbol{AD = 8}$$

$$\boldsymbol{FD = AD - AF}$$
$$\boldsymbol{FD = 8 - 2}$$
$$\boldsymbol{FD = 6}$$

There could be two answers; however, if $x = 0$, then all the points on the circle and on the lines would coincide, and that leads to an impossible diagram as described, so the value of x must be 4.

Lesson 17: Writing the Equation for a Circle

1. Use the distance formula (or Pythagorean theorem) and the definition of a circle to describe how the equation $x^2 + y^2 = 9$ defines a circle. Indicate the center point and radius of the circle.

   In the given equation $x^2 + y^2 = 9$, the values x and y, respectively, represent the coordinates of all points (x, y) that lie at a distance of 3 from the origin. This is shown using the distance formula,

   $3 = \sqrt{(x-0)^2 + (y-0)^2}$,

   or equivalently using the Pythagorean theorem,

   $(x-0)^2 + (y-0)^2 = 3^2$.

   Both cases above are equivalent to $x^2 + y^2 = 9$.

   By definition, the set of points that lie at a fixed distance $r > 0$ from a point C is called a circle with center C and radius r. In this case, the center of the circle is the origin $(0, 0)$, and the radius of the circle is 3.

   The distance formula is an equivalent variation of the Pythagorean theorem, which relates the sides of right triangles. If the vertex of one acute angle in a right triangle is the origin $(0, 0)$, and the vertex of the other acute angle of the right triangle is at the point (x, y), then the legs of the right triangle would be parallel to the x- and y-axes. The horizontal leg would have a length of x, the vertical leg would have a length of y, and the hypotenuse would have a length of 3.

2. The equation of a circle is given by $(x-1)^2 + (y-6)^2 = 16$. Trevor states that the center of the circle is the point $(-1, -6)$ and the radius of the circle is 16. Explain why you agree or disagree with Trevor.

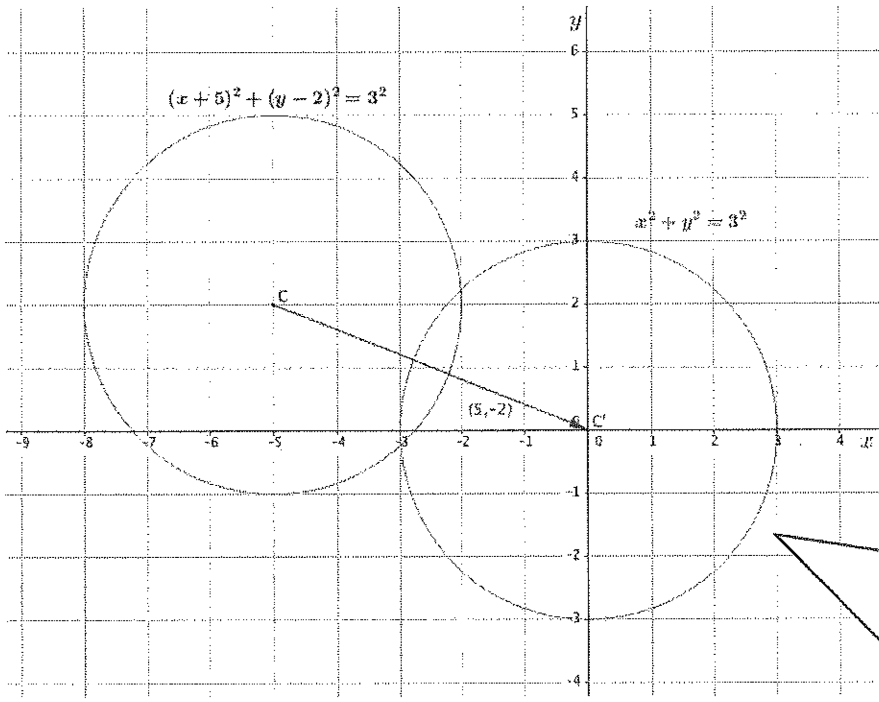
   I disagree with Trevor. He has two misunderstandings.

   First, the center of the circle is actually $(1, 6)$. The minus signs in the equation come from a translation along a vector that maps the center of the circle to the origin, which would be the vector $\langle -1, -6 \rangle$.

   His second misunderstanding is that the radius of the circle is $\sqrt{16}$, or 4. In the Pythagorean theorem, the square of the hypotenuse of a right triangle (in this case a radius of the circle) is equal to the sum of the squares of its legs. Trevor considered the length of the hypotenuse to be the square.

   I remember from the lesson that the equation of a circle is derived partly by the translation that maps the center of the circle to the origin. The translation that maps this circle's center to the origin is $\langle -1, -6 \rangle$, so the center must be at $(1, 6)$.

3. Using the grid below, show how the equation of the given circle is derived.



The equation for a circle centered at the origin is easy: $x^2 + y^2 = r^2$. The center of the given circle is not the origin, but I can translate the center to the origin to get a congruent circle.

The center of the given circle is at the point $(-5, 2)$. The radius of the circle is 3. If a congruent circle was located at the origin, its equation would be $x^2 + y^2 = 3^2$, or $x^2 + y^2 = 9$. A congruence that would map the center of the given circle to the origin would be a translation along the vector $\langle 5, -2 \rangle$. Thus, the equation of the given circle is $(x+5)^2 + (y-2)^2 = 9$.

Lesson 18: Recognizing Equations of Circles

1. The graph of quadratic equation $x^2 + y^2 - 14x + 2y = -32$ is a circle. Identify the center and radius.

$$x^2 + y^2 - 14x + 2y = -32$$
$$x^2 - 14x + y^2 + 2y = -32$$
$$x^2 - 14x + 49 + y^2 + 2y + 1 = -32 + 49 + 1$$
$$(x - 7)^2 + (y + 1)^2 = 18$$

The circle is centered at the point $(7, -1)$ and has a radius of $\sqrt{18}$, or $3\sqrt{2}$.

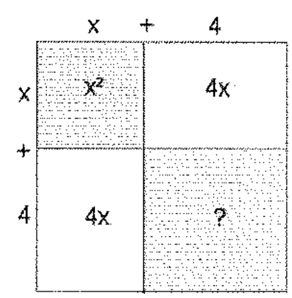
Center and radius form for the equation of a circle is $(x - a)^2 + (y - b)^2 = r^2$. I can write this equation in that form by completing the square for the expression in x and also for the expression in y.

2. The quadratic equation $x^2 + y^2 + 8x - 6y = -30$ is not a circle. Explain why.

$$x^2 + y^2 + 8x - 6y = -30$$
$$x^2 + 8x + y^2 - 6y = -30$$
$$x^2 + 8x + 16 + y^2 - 6y + 9 = -30 + 16 + 9$$
$$(x + 4)^2 + (y - 3)^2 = -5$$

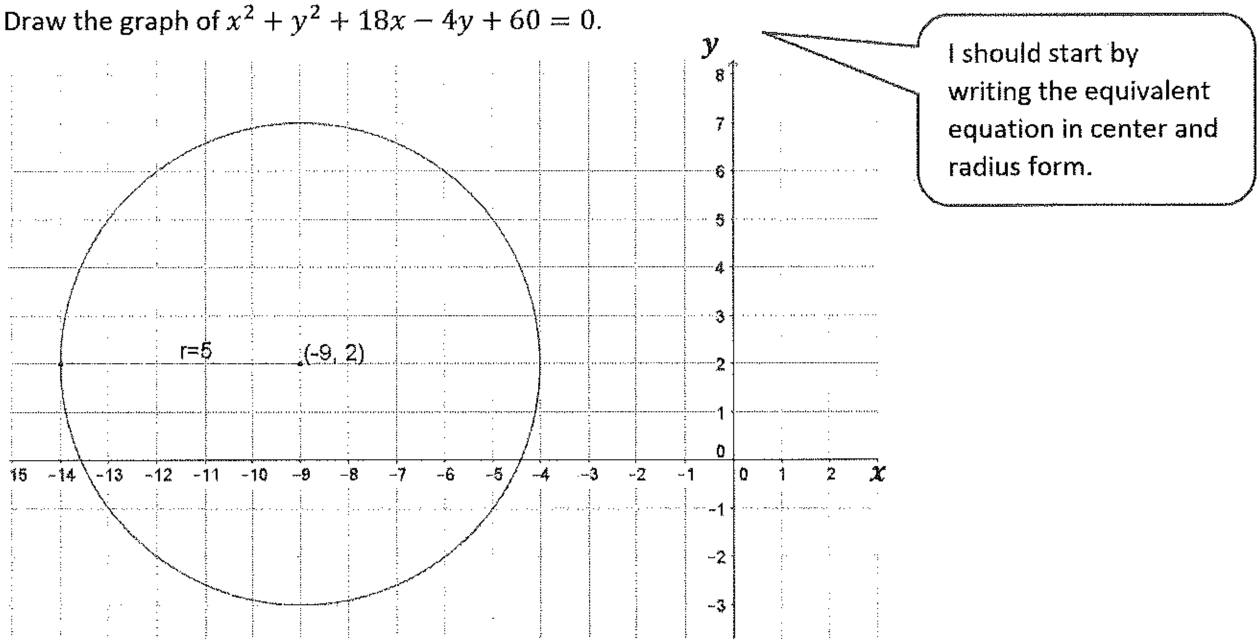
This equation does not represent a circle because the right side of the equation is a negative value, in which case the radius of the circle would have to be $\sqrt{-5}$, which is not a real number.

To complete the square, I first group the x-terms and the y-terms in the given equation. So I have $x^2 + 8x$ and $y^2 - 6y$.



The product $(x + 4)(x + 4)$ provides $x^2 + 8x + ?$, and the dimensions of the region to complete the square are $4 \cdot 4$, which is an area of 16. So I need to add 16 to both sides of the equation. Then I repeat the process for $y^2 - 6y$.

3. Draw the graph of $x^2 + y^2 + 18x - 4y + 60 = 0$.



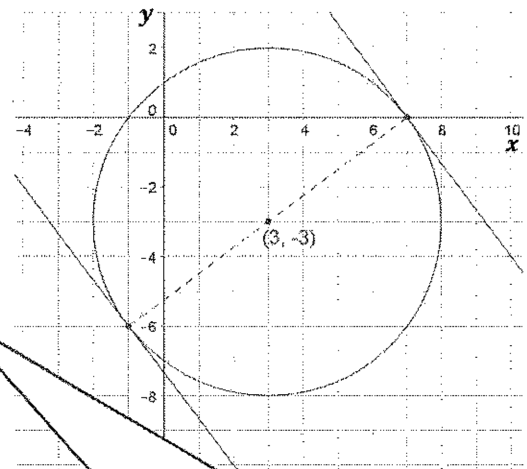
$$x^2 + y^2 + 18x - 4y + 60 = 0$$
$$x^2 + 18x + y^2 - 4y = -60$$
$$x^2 + 18x + (9)^2 + y^2 - 4y + (-2)^2 = -60 + (9)^2 + (-2)^2$$
$$(x + 9)^2 + (y - 2)^2 = -60 + 81 + 4$$
$$(x + 9)^2 + (y - 2)^2 = 25$$

Lesson 19: Equations for Tangent Lines to Circles

1. Consider the circle $(x-3)^2 + (y+3)^2 = 25$. There are two tangent lines to the circle with slopes $-\frac{4}{3}$.

   a. Find the coordinates of the points of tangency.

The circle is centered at $(3, -3)$ and has a radius of 5. The tangents to the circle must have slopes of $-\frac{4}{3}$. Tangents to a circle are perpendicular to radii of the circle at the point of tangency, so the radii to the points of tangency have slopes of $\frac{3}{4}$.



The equation of the diameter that is perpendicular to the tangents is $y + 3 = \frac{3}{4}(x-3)$, or in slope-intercept form is $y = \frac{3}{4}x - \frac{21}{4}$.

The points of tangency must be solutions to the system

$$\begin{cases} (x-3)^2 + (y+3)^2 = 25 \\ y = \frac{3}{4}x - \frac{21}{4} \end{cases}$$

$$(x-3)^2 + \left(\left(\frac{3}{4}x - \frac{21}{4}\right) + 3\right)^2 = 25$$

$$x^2 - 6x + 9 + \frac{9}{16}x^2 - \frac{54}{16}x + \frac{81}{16} = 25$$

$$\frac{25}{16}x^2 - \frac{150}{16}x + \frac{225}{16} = 25$$

$$25x^2 - 150x + 225 = 400$$

$$x^2 - 6x + 9 = 16$$

$$(x-3)^2 = 16$$

$$x - 3 = \pm 4$$

$x - 3 = 4$ *or* $x - 3 = -4$

$x = 7$ $\qquad$ $x = -1$

$y = \frac{3}{4}(7) - \frac{21}{4}$ *or* $y = \frac{3}{4}(-1) - \frac{21}{4}$

$y = 0$ $\qquad$ $y = -6$

The points of tangency are $(7, 0)$ and $(-1, -6)$.

I know that the tangent lines have to have slopes of $-\frac{4}{3}$, so the diameter of the circle that is perpendicular to the tangent lines has to have a slope that is the negative reciprocal of $-\frac{4}{3}$, which is $\frac{3}{4}$.

The graphs on the coordinate plane are solution sets to the equation of the line and the equation of the circle. I can find their intersection by solving the system of equations algebraically.

b. Find the equations of the two tangent lines in slope-intercept form.

The slope of both tangents lines is $-\frac{4}{3}$. *Using point-slope form, the equation of the tangent line on* $(-1,-6)$ *in point-slope form is*

$$y+6=-\frac{4}{3}(x+1)$$
$$y=-\frac{4}{3}(x+1)-6$$
$$y=-\frac{4}{3}x-\frac{4}{3}-6$$
$$y=-\frac{4}{3}x-\frac{22}{3}$$

> To write the equation in slope-intercept form, I need to know the slope and the y-intercept of the line. I know the slope of both lines is $-\frac{4}{3}$, so I can use point-slope form of a line and then solve for y.

The equation of the tangent line on $(7,0)$ *in point-slope form is*

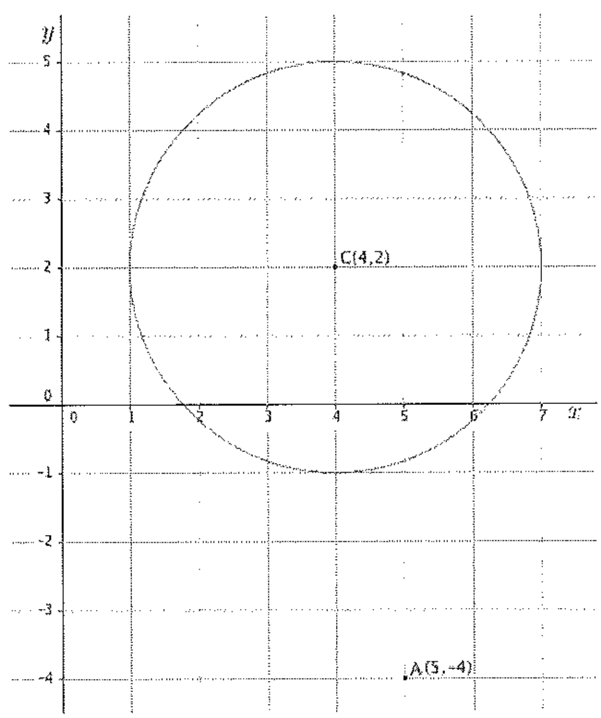
$$y=-\frac{4}{3}(x-7)$$
$$y=-\frac{4}{3}x+\frac{28}{3}$$

2. Follow the steps below to find the equations of the tangent lines to the circle with equation $(x-4)^2+(y-2)^2=9$ from point $A(5,-4)$.

   a. Using the grid to the right, graph point A and the circle described by the equation above. Identify the coordinates of center, C, and the radius of the circle.

   The center of the circle is $C(4,2)$, *and the radius is* 3.

> The equation is given in center and radius form $(x-a)^2+(y-b)^2=r^2$, so the center is (a,b) and the radius is r.



b. Calculate distance AC.

$AC = \sqrt{(5-4)^2 + (-4-2)^2}$

$AC = \sqrt{1^2 + (-6)^2}$

$AC = \sqrt{1+36}$

$AC = \sqrt{37}$

Distance AC is $\sqrt{37}$, which is approximately 6.1.

c. Use distance AC and the radius of the circle found in part (a) to calculate the distance from A to the points of tangency B_1 and B_2.

The distance $AB_1 = AB_2$ because tangents to a circle from the same point are equal in length. Tangents to a circle are also perpendicular to radii of the circle at the point of tangency, so $\triangle AB_1C$ and $\triangle AB_2C$ are both right triangles.

A tangent meets a circle at a single point and meets the radius drawn to that point of tangency at a right angle. This means I have a right triangle and I know the lengths of two of its sides.

Using the Pythagorean theorem:

$(AB_1)^2 + (B_1C)^2 = (AC)^2$

$(AB_1)^2 + 3^2 = \left(\sqrt{37}\right)^2$

$(AB_1)^2 + 9 = 37$

$(AB_1)^2 = 28$

$AB_1 = \sqrt{28}$

Points B_1 and B_2 lie at a distance of $\sqrt{28}$ from point A.

d. Find the equation of the set of points at a distance AB_1 from point A (the circle centered at A with a radius of AB_1).

The center of the circle is $A(5,-4)$, and the radius of the circle is $\sqrt{28}$, so the equation of the circle, in center and radius form is $(x-5)^2 + (y+4)^2 = 28$.

e. If points B_1 and B_2 are equidistant from C and also equidistant from A, then B_1 and B_2 lie on both circles described by the equations above. Use the equations of the circles to find the coordinates of B_1 and B_2.

$$\begin{cases}(x-4)^2+(y-2)^2=9\\(x-5)^2+(y+4)^2=28\end{cases}$$

$$\begin{cases}x^2-8x+16+y^2-4y+4=9\\x^2-10x+25+y^2+8y+16=28\end{cases}$$ *Distributive property*

$$0x^2-2x+9+0y^2+12y+12=19$$ *Elimination*

$$-2x+12y+21=19$$
$$12y=2x-2$$
$$y=\frac{1}{6}x-\frac{1}{6}$$

> The elimination method for solving systems of equations helps me eliminate the presence of x^2 and y^2 in my system. Then I am left with a linear relationship between x and y.

The tangent points are the intersection of the graph of $y=\frac{1}{6}x-\frac{1}{6}$ and the graph of $(x-4)^2+(y-2)^2=9$.

$$(x-4)^2+\left(\frac{1}{6}x-\frac{1}{6}-2\right)^2=9$$ *Substitution*

$$x^2-8x+16+\left(\frac{1}{6}x-\frac{13}{6}\right)^2=9$$
$$x^2-8x+16+\frac{1}{36}x^2-\frac{26}{36}+\frac{169}{36}=9$$
$$\frac{37}{36}x^2-\frac{314}{36}+\frac{421}{36}=0$$
$$37x^2-314+421=0$$

> This is not going to be an easy equation to factor, so I will use the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Using the quadratic formula:

$$x=\frac{-(-314)\pm\sqrt{(-314)^2-4(37)(421)}}{2(37)}$$
$$x=\frac{314\pm\sqrt{98596-62308}}{74}$$
$$x=\frac{314\pm\sqrt{36288}}{74}$$
$$x=\frac{314\pm72\sqrt{7}}{74}$$
$$x=\frac{157\pm36\sqrt{7}}{37}$$
$$x=\frac{157+36\sqrt{7}}{37} \quad \textit{or} \quad x=\frac{157-36\sqrt{7}}{37}$$

Using substitution:

$$y = \frac{1}{6}\left(\frac{157 + 36\sqrt{7}}{37}\right) - \frac{1}{6}$$
$$y = \frac{1}{6}\left(\frac{157 + 36\sqrt{7}}{37} - 1\right)$$
$$y = \frac{157 + 36\sqrt{7} - 37}{222}$$
$$y = \frac{120 + 36\sqrt{7}}{222}$$

$$y = \frac{1}{6}\left(\frac{157 - 36\sqrt{7}}{37}\right) - \frac{1}{6}$$
$$y = \frac{1}{6}\left(\frac{157 - 36\sqrt{7}}{37} - 1\right)$$
$$y = \frac{157 - 36\sqrt{7} - 37}{222}$$
$$y = \frac{120 - 36\sqrt{7}}{222}$$

The tangents to the circle from point A meet the circle at $B_1\left(\frac{157 + 36\sqrt{7}}{37}, \frac{120 + 36\sqrt{7}}{222}\right)$ and $B_2\left(\frac{157 - 36\sqrt{7}}{37}, \frac{120 - 36\sqrt{7}}{222}\right)$, which are approximately $(6.8, 1)$ and $(1.7, 0.1)$, respectively.

f. Use the points of tangency that you found to determine the equation of one of the two tangent lines to the circle.

$A(5, -4)$ and $B_2\left(\frac{157 - 36\sqrt{7}}{37}, \frac{120 - 36\sqrt{7}}{222}\right)$

> To find an equation of a line, I need a point and the slope of the line. I know the line includes $A(5, -4)$, so I just need to calculate its slope.

$$\text{Slope}_{\overline{AB_2}} = \frac{\frac{120 - 36\sqrt{7}}{222} - (-4)}{\frac{157 - 36\sqrt{7}}{37} - 5}$$

$$\text{Slope}_{\overline{AB_2}} = \frac{\frac{120 - 36\sqrt{7}}{222} + \frac{888}{222}}{\frac{157 - 36\sqrt{7}}{37} - \frac{185}{37}}$$

$$\text{Slope}_{\overline{AB_2}} = \frac{\frac{1008 - 36\sqrt{7}}{222}}{\frac{-28 - 36\sqrt{7}}{37}}$$

$$\text{Slope}_{\overline{AB_2}} = \frac{1008 - 36\sqrt{7}}{222} \cdot \frac{37}{-28 - 36\sqrt{7}}$$

$$\text{Slope}_{\overline{AB_2}} = \frac{1008 - 36\sqrt{7}}{-168 - 216\sqrt{7}}$$

In point-slope form, the equation of the tangent line through A and B_2 is

$$y + 4 = -\frac{1008 - 36\sqrt{7}}{168 + 216\sqrt{7}}(x - 5),$$

> If I factor out (-1) from the denominator, I can write the value of the expression itself as a negative value.

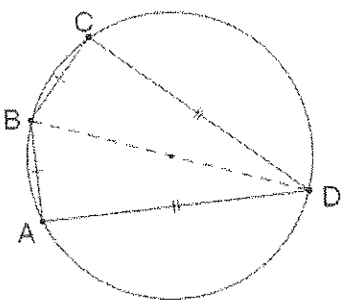
which is approximately $y + 4 = -1.2(x - 5)$.

EUREKA MATH

Lesson 20: Cyclic Quadrilaterals

A kite is a quadrilateral in which two adjacent sides have equal length and the remaining two sides also have equal length.

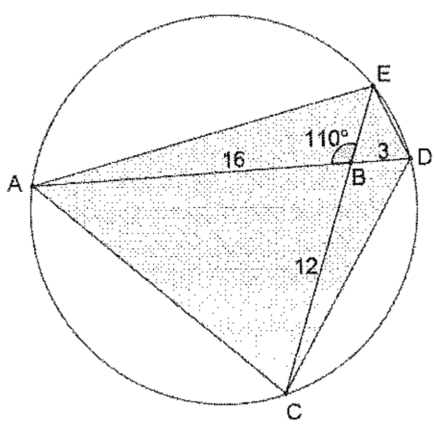
1. Kite $ABCD$, with $AB = CB$ and $CD = AD$, is cyclic. What must be true about the measures of angles A and C? Explain.



Opposite angles in a cyclic quadrilateral are supplementary. In the kite described, opposite angles A and C must be equal in measure as they are corresponding angles in congruent triangles ABD and CBD. If the kite is cyclic and angles A and C are also congruent, then angles A and C are both right angles.

I know that angles A and C are congruent because if I draw diagonal $\overline{BD}$, then the kite is composed of two congruent triangles by SSS.

2. Find the area of cyclic quadrilateral $ACDE$.



Using the two-chord power rule,

$$AB \cdot BD = CB \cdot BE$$
$$16 \cdot 3 = 12 \cdot BE$$
$$BE = 4$$

Angle DBE is supplementary to angle ABE since the angles are on a line, so $m\angle DBE = 70°$.

The area of a cyclic quadrilateral is equal to one half the product of the lengths of its diagonals times the sine of the acute angle formed by them.

To find the area of the cyclic quadrilateral, I need the lengths of its diagonals and the degree measure of the acute angle formed by the diagonals.

$$\text{Area} = \frac{1}{2}(AD)(CE) \cdot \sin 70$$

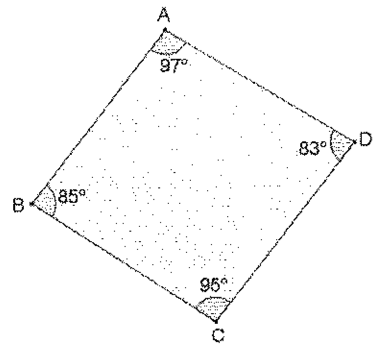
$$\text{Area} = \frac{1}{2}(19)(16) \cdot \sin 70$$

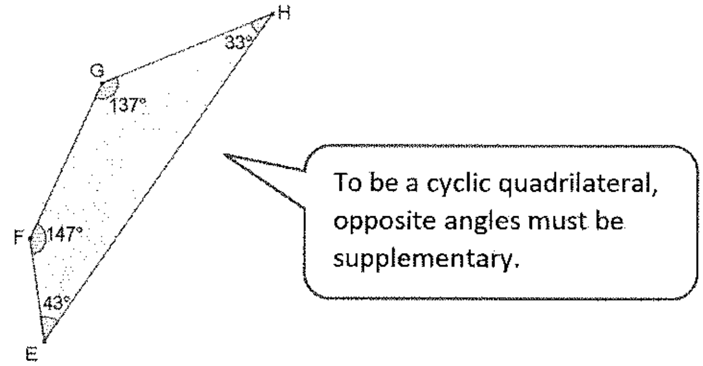
$$\text{Area} = 152 \cdot \sin 70$$

$$\text{Area} \approx 142.8$$

The area of cyclic quadrilateral $ACDE$ is approximately 142.8 square units.

3. One of the quadrilaterals shown below is cyclic and the other is not. Explain why each is or is not cyclic. Construct the circle on the one that is cyclic.

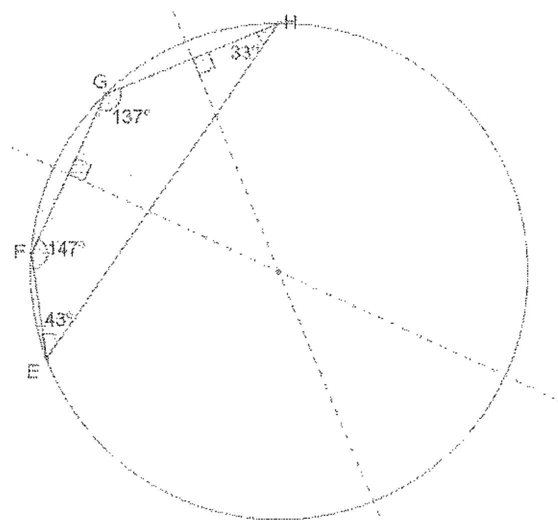




Quadrilateral $ABCD$ is not cyclic because its opposite angles are not supplementary.

Quadrilateral $EFGH$ is cyclic because its opposite angles are supplementary.

First, construct the perpendicular bisectors of any two sides of the quadrilateral. The sides are chords of the circumscribed circle, and the center of the circle lies on the perpendicular bisector of any chord in the circle. The center of the circle is the intersection point of the two perpendicular bisectors.



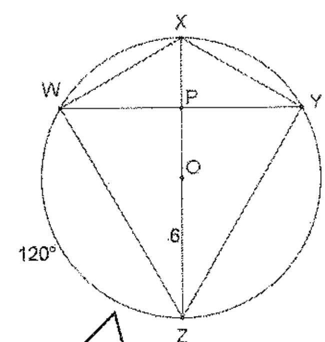
Lesson 21: Ptolemy's Theorem

1. Describe in your own words what Ptolemy's theorem claims about a cyclic quadrilateral.

If a quadrilateral is cyclic, then the product of the lengths of its diagonals is equal to the sum of the products of the lengths of its opposite sides.

> In Ptolemy's theorem, the relationship of the sides of a cyclic quadrilateral $ABCD$ is $AC \cdot BD = AB \cdot CD + BC \cdot AD$. $\overline{AC}$ and $\overline{BD}$ are diagonals of the quadrilateral. $\overline{AB}$ and $\overline{CD}$ are a pair of opposite sides of the quadrilateral, and $\overline{BC}$ and $\overline{AD}$ are the other pair of opposite sides.

2. A circle with radius of 6 circumscribes kite $WXYZ$, $m\widehat{WZ} = m\widehat{YZ} = 120°$. Use Ptolemy's theorem to find the perimeter of kite $WXYZ$.



$m\angle WXZ = m\angle YXZ = 60°$	*Inscribed angle theorem*
$m\widehat{WX} = m\widehat{XY} = 60°$	*Arc measures of a semicircular arc total 180°.*
$m\angle XOW = m\angle XOY = 60°$	*Arc measure is equal to that of its central angle.*
$m\angle OWX = m\angle OYX = 60°$	*Angle measures of a triangle sum to 180°.*

$\triangle XOW$ and $\triangle XOY$ are equilateral because all angle measures are 60°.

$OX = WX = 6$ $OX = XY = 6$	*Sides of an equilateral triangle are equal in length.*
$\overline{WY} \perp \overline{XZ}$	*Diagonals of a kite are perpendicular.*
$PX = 3$, $PW = PY = 3\sqrt{3}$	*Pythagorean theorem (30°–60°–90° triangle)*
$WZ = YZ$	*Congruent consecutive sides of a kite*
$XZ \cdot WY = WX \cdot YZ + XY \cdot WZ$	*Ptolemy's theorem*
$12 \cdot 6\sqrt{3} = 6 \cdot YZ + 6 \cdot WZ$	*Substitution*
$72\sqrt{3} = 12 \cdot YZ$	*Substitution, $WZ = YZ$*
$YZ = 6\sqrt{3}$	

> To use Ptolemy's theorem, I need to relate the lengths of the diagonals of the kite to the lengths of the sides of the kite. I know that the diagonals of a kite are perpendicular, so I have several right triangles on which I can use trigonometry or the Pythagorean theorem.

$YZ = WZ = 6\sqrt{3}$ *Substitution*

Perimeter of kite $WXYZ$:

$\text{Perimeter} = WX + XY + YZ + WZ$

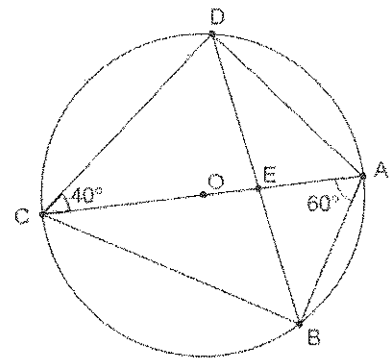
$\text{Perimeter} = 6 + 6 + 6\sqrt{3} + 6\sqrt{3}$

$\text{Perimeter} = 12 + 12\sqrt{3}$

$\text{Perimeter} \approx 32.8$

The perimeter of kite $WXYZ$ is approximately 32.8.

3. Given cyclic quadrilateral $ABCD$, with diameter $\overline{CA}$ and $CA = 16$, what is the length of chord $\overline{DB}$?



$\overline{CA}$ is a diameter, so $\triangle CDA$ and $\triangle CBA$ are right triangles. By the angle sum of a triangle, $m\angle DAC = 50°$ and $m\angle BCA = 30°$.

By the Pythagorean theorem ($30°$–$60°$–$90°$ triangle), $AB = 8$ and $BC = 8\sqrt{3}$.

Using right triangle trigonometry:

$$\sin 50 = \frac{CD}{16}$$
$$CD = 16 \sin 50$$

$$\cos 50 = \frac{AD}{16}$$
$$AD = 16 \cos 50$$

Using Ptolemy's theorem:

$$16 \cdot DB = 8 \cdot 16 \sin 50 + 8\sqrt{3} \cdot 16 \cos 50$$
$$16 \cdot DB = 8 \cdot 16 \cdot (\sin 50 + \sqrt{3} \cos 50)$$
$$DB = 8(\sin 50 + \sqrt{3} \cos 50)$$
$$DB \approx 15.0$$

The length of chord $\overline{DB}$ is approximately 15.0.

I know that inscribed angles subtended by a diameter must be right angles, and I also know the length of the hypotenuse of right triangles CDA and CBA, so I can use right triangle trigonometry to find the lengths of the legs of the right triangles.