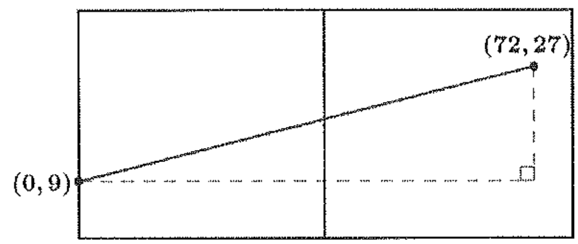
Homework Helpers

Geometry Module 4

Lesson 1: Searching a Region in the Plane

1. A tennis player hits the ball from position $(0, 9)$ directly at her opponent at position $(72, 27)$. The ball reaches the opponent after 0.5 second. The tennis court is measured in feet.

   a. Plot the points representing the two players, and draw the segment connecting the points.



   b. What was the change in the x-coordinate?

   Since $72 - 0 = 72$, the change in the x-coordinate is 72.

   c. What was the change in the y-coordinate?

   Since $27 - 9 = 18$, the change in the y-coordinate is 18.

   d. What is the ratio of the change in y to the change in x?

   The ratio of change is $18:72$, which reduces to $1:4$.

   e. How far did the ball travel between the two players?

   I need to find the length of the hypotenuse of a triangle with legs of length 18 and 72.

   The distance, d, between the two players can be found using the Pythagorean theorem.

$$d = \sqrt{18^2 + 72^2}$$
$$d = \sqrt{5508}$$
$$d \approx 74.22$$

   The ball travelled approximately 74.22 feet.

   f. What was the speed of the ball?

   The ball travelled approximately 74.22 feet in 0.5 second, and $\frac{74.22}{0.5} = 148.44$, so the speed of the ball was approximately 148.44 feet per second.

   I know that speed is distance divided by time.

2. You want to program a robot vacuum cleaner to clean a vacant 44 ft. by 30 ft. room. The robot can travel in a straight line and moves at a constant speed. The vacuum is at position $(12, 6)$ after 4 seconds and at position $(36, 16)$ after 12 seconds.

   a. How far does the robot travel over 8 seconds?

$$d = \sqrt{(36-12)^2 + (16-6)^2}$$
$$d = \sqrt{576 + 100}$$
$$d = \sqrt{676}$$
$$d = 26$$

I can use the distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. In this problem, $(x_1, y_1) = (12, 6)$ and $(x_2, y_2) = (36, 16)$.

The robot travels 26 feet during this 8 seconds.

   b. What is the constant speed of the robot?

Since the robot traveled 26 feet in 8 seconds, its speed is 3.25 feet per second.

Speed is distance divided by time, so the speed is $\frac{26}{8}$ feet per second.

   c. What is the ratio of the change in the x-coordinate to the change in the y-coordinate?

The change in the x-cooordinate is $36 - 12$, which is 24. The change in the y-coordinate is $16 - 6$, which is 10. The ratio of change in the x-coordinate to the change in the y-coordinate is $24:10$, which reduces to $12:5$.

   d. Where did the robot start?

In 8 seconds, the robot travels 24 units horizontally and 10 units vertically. Then, in the first 4 seconds, it travels 12 units horizontally and 5 units vertically because it is traveling at a constant speed. The original position is 12 units to the left and 5 units down from $(12, 6)$. The original position is $(0, 1)$.

   e. Where will the robot be when 14 seconds have elapsed?

After 14 seconds, the robot will be 6 units to the right and 2.5 units up from $(36, 16)$, so it will be at position $(42, 18.5)$.

f. At what location will the robot hit the wall?

The robot will hit the wall when it travels an additional 2 *feet horizontally from* $(42, 18.5)$. *When it travels* 2 *feet horizontally, it travels* $\frac{5}{6}$ *of a foot vertically. The robot will hit the wall at the approximate location* $(44, 19.33)$.

I know from part (c) that the ratio of horizontal to vertical movement is 12: 5.

g. At what time will the robot hit the wall?

The robot travels from the starting position $(0, 1)$ *to the approximate position* $(44, 19.33)$.

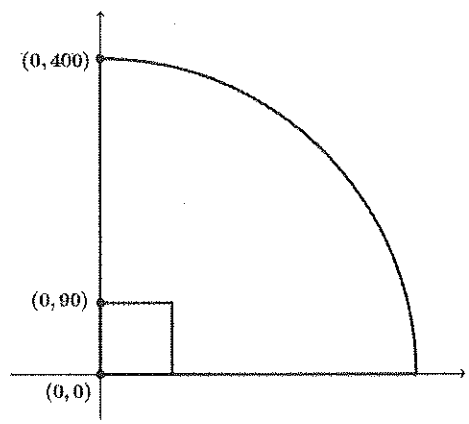
$$d \approx \sqrt{(44-0)^2 + (19.33-1)^2}$$
$$d \approx \sqrt{1936 + 335.99}$$
$$d \approx 47.67$$

The robot travels approximately 47.67 *feet at a speed of* 3.25 *feet per second, so it takes approximately* 155 *seconds for the robot to reach the far wall.*

3. A baseball player hits a ball at home plate at position $(0, 0)$. It travels at a constant speed across third base at position $(0, 90)$ in 0.75 second.

a. What was the speed of the ball?

The ball traveled 90 *feet in* 0.75 *second, so its speed was* $\frac{90}{0.75}$ *feet per second, which is equivalent to* 120 *feet per second.*

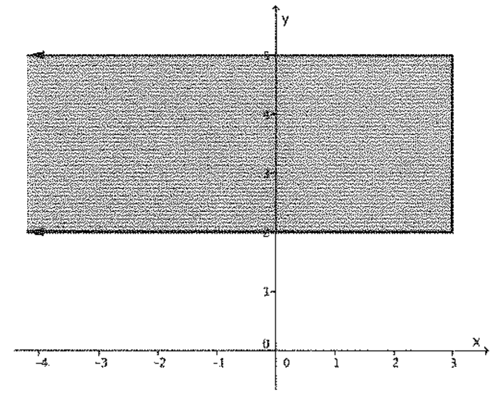


b. When will it hit the outfield fence at position $(0, 400)$? Explain how you know.

The ball needs to travel 400 *feet at a speed of* 120 *feet per second. Since* $\frac{400}{120} \approx 3.33$, *the ball will hit the outfield fence after approximately* 3.33 *seconds.*

Lesson 2: Finding Systems of Inequalities That Describe Triangular and Rectangular Regions

1. Consider the region shown.



   a. How many half-planes intersect to form this region?

   Three half-planes intersect to form this region.

   One half-plane lies to the left of the right edge, another half plane lies below the top edge, and the third half-plane lies above the bottom edge.

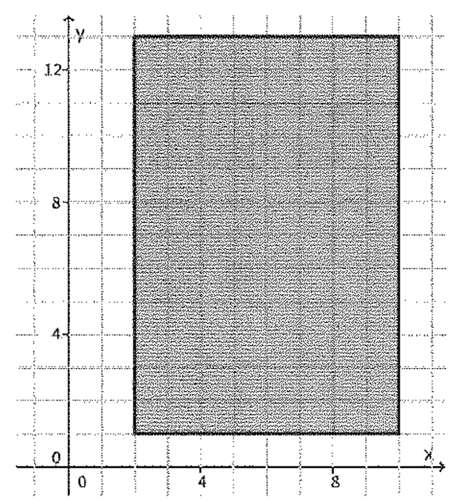
   b. Identify three points on the boundary of the region.

   For example, $(-2, 2)$, $(3, 4)$, and $(-3, 5)$ all lie on the boundary. There are infinitely many other correct answers.

   c. Describe the region in words.

   The region is the set of points that are between or on the lines $y = 2$ and $y = 5$ and to the left of or on the line $x = 3$.

2. Region R is shown to the right.



   a. Write the coordinates of the vertices of region R.

   The vertices have coordinates $(2, 1)$, $(10, 1)$, $(10, 13)$, and $(2, 13)$.

   b. Describe the region using inequalities.

   The points (x, y) in region R satisfy $2 \le x \le 10$ and $1 \le y \le 13$.

   I can also use set notation.
   $\{(x, y) \mid 2 \le x \le 10, 1 \le y \le 13\}$

c. What is the length of a diagonal of the rectangle bounding region R?

The length of a diagonal is the distance, d, from point $(2, 1)$ to $(10, 13)$.

$$d = \sqrt{(10-2)^2 + (13-1)^2}$$
$$d = \sqrt{64 + 144}$$
$$d = \sqrt{208}$$
$$d \approx 14.42$$

The diagonal is approximately 14.42 units long.

I remember that the distance, d, between points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

d. Give coordinates of a point that is both in region R and on a diagonal.

The ratio of the change in the y-coordinate to the change in the x-coordinate is $3:2$. One point that is in both region R and on a diagonal is $(4, 4)$. There are other correct responses.

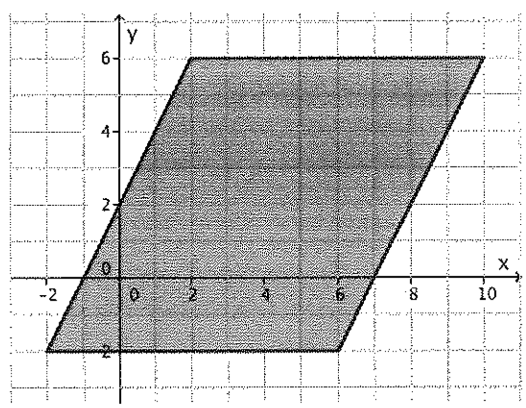
If I add a multiple of 2 to the x-coordinate of $(2, 1)$ and the same multiple of 3 to the y-coordinate of $(2, 1)$, I will get a point on the line through the diagonal from $(2, 1)$ to $(10, 13)$.

e. Write the equation of a line that passes through region R.

There are infinitely many lines that pass through region R. One such line has equation $y = x$.

To find the equation of a line that passes through R, I can choose two points on different sides of the boundary and find the equation of the line that connects them. The line $y = x$ connects points $(2, 2)$ and $(10, 10)$.

3. Let T be the region bounded by the parallelogram shown.



a. Write the system of inequalities that describe region T.

The lines that bound the parallelogram are $y = -2$, $y = 2x - 14$, $y = 6$, and $y = 2x + 2$.

I know the equation of a horizontal line has the form $y = k$, and I know how to find the equation of a line through two points.

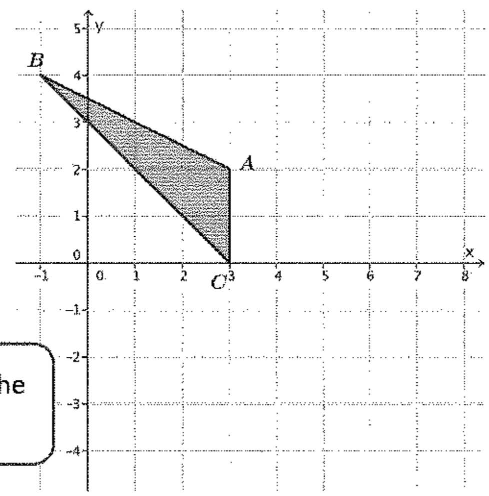
The inequalities that describe region T are $y \geq -2$, $y \geq 2x - 14$, $y \leq 6$, and $y \leq 2x + 2$.

Once I have the equations of the lines bounding region T, I can determine which inequality to use by testing with the point $(0, 0)$, which is in T.

b. Translate the parallelogram 5 units to the left and 3 units up. Write the new system of inequalities that describes the translated region.

The new parallelogram has vertices $(-7, 1)$, $(1, 1)$, $(5, 9)$, *and* $(-3, 9)$. *The resulting inequalities are* $y \geq 1$, $y \geq 2x - 1$, $y \leq 9$, *and* $y \leq 2x + 15$.

4. Consider the triangular region shown with vertices $A(3, 2)$, $B(-1, 4)$, and $C(3, 0)$.



a. Write the system of inequalities that describes the region enclosed by the triangle.

$x \leq 3$

> The equation for $\overleftrightarrow{AC}$ is $x = 3$, and the interior of the triangle is to the left of this line.

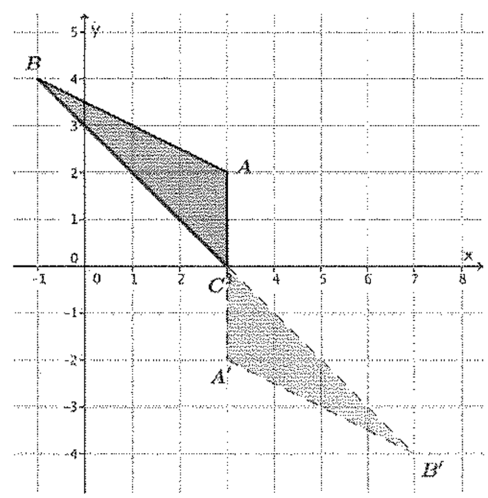
$y \geq -x + 3$

> The equation for $\overleftrightarrow{BC}$ is $y = -x + 3$, and the interior of the triangle is above this line.

$y \leq -\frac{1}{2}x + \frac{7}{2}$

> The equation for $\overleftrightarrow{AB}$ is $y = -\frac{1}{2}x + \frac{7}{2}$, and the interior of the triangle is below this line.

b. Rotate the region by $180°$ about point C. How will this change the coordinates of the vertices?



The rotated triangle is shown to the right. The image of point C is point C, so the coordinates do not change. The coordinates of the image of point A are $(3, -2)$, *and the coordinates of the image of point B are* $(7, -4)$.

c. Write the system of inequalities that describes the region enclosed by the rotated triangle.

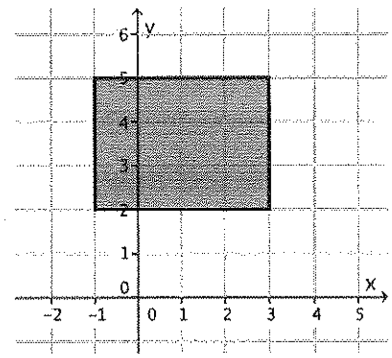
$x \geq 3$
$y \leq -x + 3$
$y \geq -\frac{1}{2}x - \frac{1}{2}$

> When I rotate by $180°$ around point C, the equations of lines through C do not change.

Lesson 3: Lines That Pass Through Regions

1. Consider the rectangular region shown.

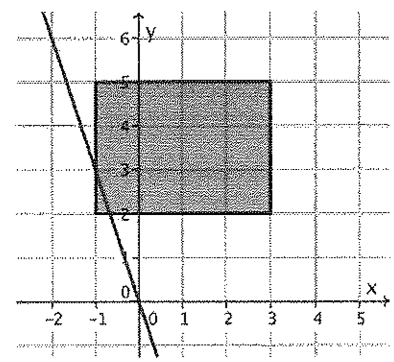
   a. Which boundary points does a line through the origin with slope -3 intersect? What is the length of the segment within the region along this line?



The region shown is defined by the inequalities $-1 \leq x \leq 3$ and $2 \leq y \leq 5$. The line through the origin with slope -3 has equation $y = -3x$. The lines that form the left and lower boundaries of the rectangle have equations $x = -1$ and $y = 2$.

The graph of the line with the rectangle shows that the line intersects the left edge and the lower edge.

> A graph of the region and the line together will allow me to see which edges of the rectangle are intersected by the line.



On the left edge:

$$y = -3x$$
$$x = -1$$

So $y = -3(-1) = 3$, and the line intersects the left edge at the point $(-1, 3)$.

On the lower edge:

$$y = -3x$$
$$y = 2$$

So $-3x = 2$, and $x = -\frac{2}{3}$. The line intersects the lower edge at the point $\left(-\frac{2}{3}, 2\right)$.
The length of this segment can be calculated as follows:

$$d = \sqrt{\left(-1 - \left(-\frac{2}{3}\right)\right)^2 + (3-2)^2}$$

$$d = \sqrt{\left(-\frac{1}{3}\right)^2 + (1)^2}$$

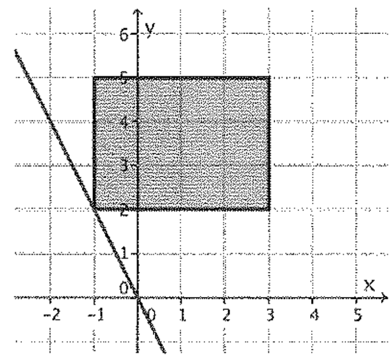
$$d = \sqrt{\frac{10}{9}}$$

$$d \approx 1.05$$

The length of the segment within the region is approximately 1.05 units long.

b. Which boundary points does a line through the origin with slope -2 intersect?

The graph of the line together with the rectangle shows that the line appears to intersect the rectangle at the lower left corner. The equation of the line is $y = -2x$, and the equation of the left edge of the rectangle is $x = -1$.



On the left edge:

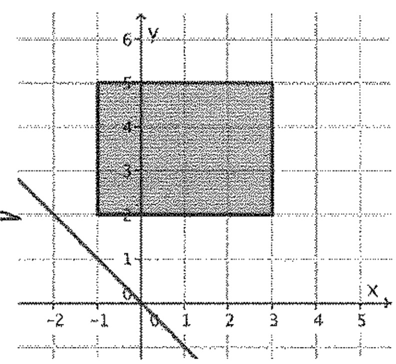
$$y = -2x$$
$$x = -1$$

So $y = -2(-1) = 2$, and the line intersects the left edge at the point $(-1, 2)$, which is the lower left corner of the rectangle.

c. Which boundary points does a line through the origin with slope -1 intersect?

The graph of the line together with the rectangle shows that the line does not appear to intersect the rectangle at all. The equation of the line is $y = -x$, and the equation of the left edge of the rectangle is $x = -1$.

A graph of the region and the line together shows that the line does not intersect the rectangular region.



On the left edge:

$$y = -x$$
$$x = -1$$

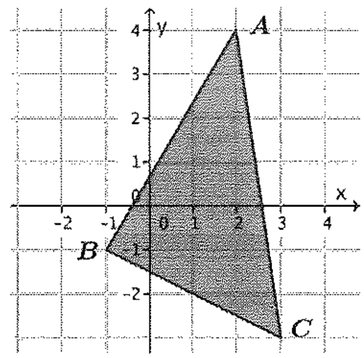
So $y = -(-1) = 1$, and the line $y = -x$ intersects the line $x = -1$ at the point $(-1, 1)$, which is outside of the rectangle.

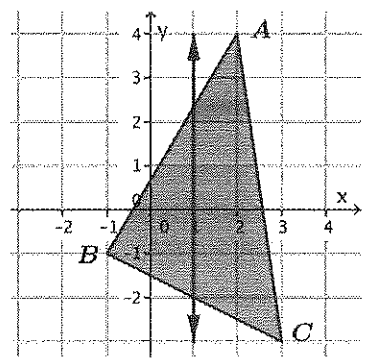
The line through the origin with slope -1 does not intersect the boundary of the rectangular region.

2. Consider the triangular region in the plane given by the triangle with vertices $A(2, 4)$, $B(-1, -1)$, and $C(3, -3)$.

   a. The vertical line with equation $x = 1$ intersects this region. What are the coordinates of the two boundary points it intersects? What is the length of the vertical segment along this line within the region?

I remember that the slope of the line between two points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$. In this case, the two points are $(2, 4)$ and $(-1, -1)$.





The line through $\overline{AB}$ has slope $\frac{5}{3}$, so it has equation $y - 4 = \frac{5}{3}(x - 2)$. An equivalent representation of this line in slope-intercept form is $y = \frac{5}{3}x + \frac{2}{3}$. When $x = 1$, this gives $y = \frac{5}{3}(1) + \frac{2}{3} = \frac{7}{3}$, so the vertical line $x = 1$ intersects $\overline{AB}$ at $\left(1, \frac{7}{3}\right)$.

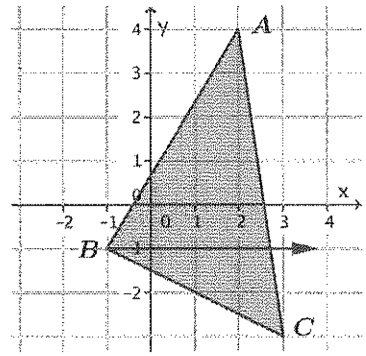
The line through $\overline{BC}$ has slope $-\frac{1}{2}$, so it has equation $y + 1 = -\frac{1}{2}(x + 1)$. The equation of this line in slope-intercept form, is $y = -\frac{1}{2}x - \frac{3}{2}$. When $x = 1$, this gives $y = -\frac{1}{2}(1) - \frac{3}{2} = -2$, so the vertical line $x = 1$ intersects $\overline{BC}$ at $(1, -2)$.

The length of the vertical segment along the line $x = 1$ in this region is $\frac{13}{3}$, which is approximately 4.3.

I need to find the distance between the points $\left(1, \frac{7}{3}\right)$ and $(1, -2)$. Since these points have the same x-coordinate, the distance is the difference of the y-coordinates.

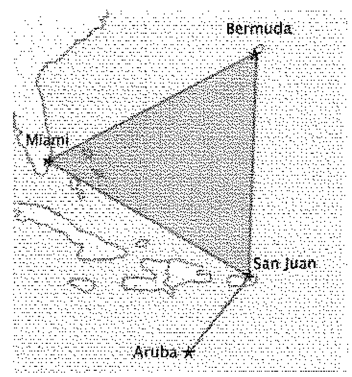
b. A robot starts at vertex B and moves horizontally to the right at a constant speed of 0.2 unit per second. When will it exit the triangular region?

The robot will exit the triangular region at the intersection of $\overline{AC}$ and the line with equation $y = -1$. The slope of the line through $\overline{AC}$ is $\frac{-3-4}{3-2}$, which is -7. Then the equation of the line through $\overline{AC}$ is $y + 3 = -7(x - 3)$, which is equivalent to $y = -7x + 18$. When $y = -1$, we can solve $-1 = -7x + 18$ to find that $x = \frac{19}{7}$. Thus, the robot will travel $\frac{26}{7}$ units within the triangle from $(-1, -1)$ to $\left(\frac{19}{7}, -1\right)$. Since the robot travels at a speed of 0.2 unit per second, this will take the robot approximately 18.57 seconds.



I can divide distance by speed to find time, and $\frac{\frac{26}{7}}{0.2} \approx 18.57$.

3. The Bermuda Triangle is defined as the triangle with vertices in Miami, San Juan, and the island of Bermuda. Before sailing in this region, Steven set up a map of the area, using the island of Aruba as the origin for his coordinate system and measuring distances in miles.



Location	Coordinates
Aruba	$(0, 0)$
San Juan	$(280, 350)$
Bermuda	$(330, 1440)$
Miami	$(-700, 910)$

a. Steven plans to sail from Aruba to San Juan, then to Bermuda, then to Miami, and back to Aruba. Approximately how far will he travel?

Distance from Aruba to San Juan: $d_1 = \sqrt{(280-0)^2 + (350-0)^2} \approx 448.2$

Distance from San Juan to Bermuda: $d_2 = \sqrt{(330-280)^2 + (1440-350)^2} \approx 1091.1$

Distance from Bermuda to Miami: $d_3 = \sqrt{(-700-330)^2 + (910-1440)^2} \approx 1158.4$

Distance from Miami to San Juan: $d_4 = \sqrt{(280-(-700))^2 + (350-910)^2} \approx 1128.7$

Distance from San Juan to Aruba: $d_5 = d_1 \approx 448.2$

The total distance Steven travels is approximately $4,275$ miles.

After finding the individual distances between ports, I need to sum them up: $d_1 + d_2 + d_3 + d_4 + d_5 \approx 4274.6$.

b. Connie is sailing due west from Miami at a constant speed of 20 miles per hour. According to Steven's map, how long will she sail before she leaves the Bermuda Triangle?

Connie will travel on the line $y = 910$. The slope of the line connecting San Juan and Bermuda is $\frac{350-1440}{280-330}$, which simplifies to 21.8. The equation of this western edge of the Bermuda Triangle is then $y - 350 = 21.8(x - 280)$, which simplifies to $y = 21.8x - 5754$.

$$910 = 21.8x - 5754$$
$$6664 = 21.8x$$
$$306 \approx x$$

I need to find the intersection of the lines $y = 910$ and $y = 21.8x - 5754$. I only need the x-coordinate since I know $y = 910$.

Because $306 - (-700) = 1006$, Connie will sail approximately $1,006$ miles at 20 miles per hour. She will leave the Bermuda Triangle after sailing for approximately 50.3 hours.

Lesson 4: Designing a Search Robot to Find a Beacon

1. Find the new coordinates of point $(5, -2)$ if it is rotated:
   a. $90°$ counterclockwise around the origin

      $(2, 5)$

      > I know that rotating a point (a, b) $90°$ counterclockwise around the origin produces the point $(-b, a)$. In this case, $a = 5$ and $b = -2$.

   b. $180°$ counterclockwise around the origin

      $(-5, 2)$

      > I know that rotating a point (a, b) $180°$ around the origin produces the point $(-a, -b)$.

   c. $270°$ counterclockwise around the origin

      $(-2, -5)$

      > Rotating $270°$ counterclockwise has the same result as rotating $90°$ clockwise. I know that rotating a point (a, b) $90°$ clockwise around the origin produces the point $(b, -a)$.

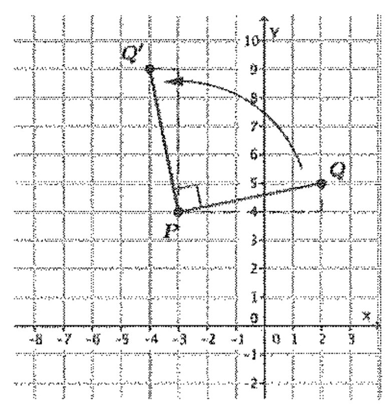
2. Line segment PQ connects points $P(-3, 4)$ and $Q(2, 5)$.
   a. What is the slope of the original segment PQ?

      Slope: $\frac{5-4}{2-(-3)} = \frac{1}{5}$

   b. Where does point Q land if the segment is rotated $90°$ counterclockwise about P? What is the slope of the rotated segment?

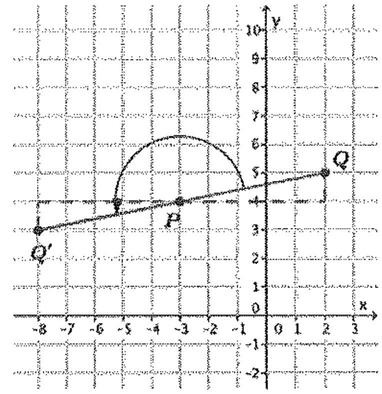
      The image point Q' has coordinates $(-4, 9)$.

      > To get from P to Q, I move 5 units to the right and 1 unit up. The picture helps me see that to get from P to the rotated point Q', I need to move 5 units up and 1 unit to the left. Then the coordinates of Q' are $(-3 - 1, 4 + 5)$.



      Slope of rotated segment PQ': $\frac{9-4}{-4-(-3)} = -5$

c. Where does point Q land if the segment is rotated 180° counterclockwise about P? What is the slope of the rotated segment?

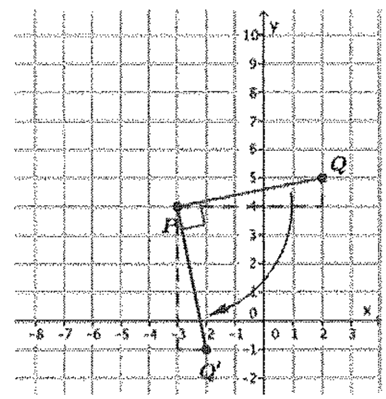


The image point Q' has coordinates $(-8, 3)$.

> To get from P to Q, I move 5 units to the right and 1 unit up. To get from P to Q', I need to move 5 units to the left and 1 unit down. Then the coordinates of Q' are $(-3-5, 4-1)$.

Slope of rotated segment PQ': $\frac{4-3}{-3-(-8)} = \frac{4-3}{-3+8} = \frac{1}{5}$

d. Where does point Q land if the segment is rotated 90° clockwise about P? What is the slope of the rotated segment?

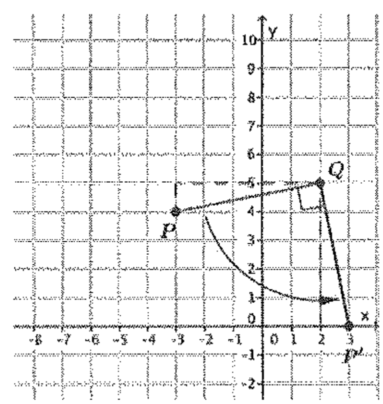


The image point Q' has coordinates $(-2, -1)$.

> To get from P to Q, I move 5 units to the right and 1 unit up. The picture helps me see that to get from P to the rotated point Q', I need to move 1 unit to the right and 5 units down. Then the coordinates of Q' are $(-3+1, 4-5)$.

Slope of rotated segment PQ': $\frac{4-(-1)}{-3-(-2)} = \frac{5}{-1} = -5$

e. Where does point P land if the segment is rotated 90° counterclockwise about Q? What is the slope of the rotated segment?



The image point P' has coordinates $(3, 0)$.

> To get from Q to P, I move 5 units to the left and 1 unit down. The picture helps me see that to get from Q to the rotated point P', I need to move 5 units down and 1 unit to the right. Then the coordinates of P' are $(2+1, 5-5)$.

Slope of rotated segment QP': $\frac{5-0}{2-3} = -5$

3. If line ℓ has slope -10, what is the slope of a line perpendicular to ℓ?

   Any line perpendicular to ℓ will have slope $\frac{1}{10}$.

   I know that the slopes of perpendicular lines are negative reciprocals of each other.

4. Line ℓ passes through the origin and has a slope of $\frac{2}{5}$.

   a. What is the slope of the line perpendicular to ℓ that passes through the origin?

      Any line perpendicular to ℓ has slope $-\frac{5}{2}$.

   b. Could a line through the origin that passes through the point $(5, -2)$ be perpendicular to ℓ?

      The line through the origin that passes through $(5, -2)$ has slope $-\frac{2}{5}$, so this line is not perpendicular to ℓ.

Lesson 5: Criterion for Perpendicularity

1. Use the converse of the Pythagorean theorem to determine whether $\overline{AC}$ is perpendicular to $\overline{AB}$ for points $A(1,1)$, $B(-2,3)$, and $C(4,5)$.

$$AB = \sqrt{(3-1)^2 + (-2-1)^2} = \sqrt{4+9} = \sqrt{13}$$
$$AC = \sqrt{(5-1)^2 + (4-1)^2} = \sqrt{16+9} = \sqrt{25} = 5$$
$$BC = \sqrt{(5-3)^2 + \left(4-(-2)\right)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$$

> The converse of the Pythagorean theorem states that if triangle ABC has sides of length a, b, and c, and $a^2 + b^2 = c^2$, then the triangle is a right triangle. Since $\overline{BC}$ is the longest side of the triangle, it would be the hypotenuse.

Then $(BC)^2 = 40$ and $(AB)^2 + (AC)^2 = 13 + 25 = 38$. Since $(AB)^2 + (AC)^2 \neq (BC)^2$, $\triangle ABC$ is not a right triangle, and $\overline{AC}$ is not perpendicular to $\overline{AB}$.

2. Use the general formula for perpendicularity of segments through the origin to determine if $\overline{OA}$ and $\overline{OB}$ are perpendicular for points $O(0,0)$, $A(-2,5)$, and $B(-5,2)$.

$$\begin{aligned} a_1b_1 + a_2b_2 &= (-2)(-5) + (5)(2) \\ &= 10 + 10 \\ &= 20 \\ &\neq 0 \end{aligned}$$

> I know that for two points $A(a_1, a_2)$ and $B(b_1, b_2)$ and the origin $O(0,0)$, $\overline{OA}$ and $\overline{OB}$ are perpendicular if $a_1b_1 + a_2b_2 = 0$.

Since $a_1b_1 + a_2b_2 \neq 0$, $\overline{OA}$ and $\overline{OB}$ are not perpendicular.

3. For points $O(0,0)$, $P(10,2)$, and $Q(-1,5)$, $\overline{OP}$ and $\overline{OQ}$ are perpendicular.

   a. Will the images of the segments be perpendicular if the three points O, P, and Q are rotated by $45°$ counterclockwise?

   Yes. Rotation preserves angles, so rotating both segments by the same amount will not change the angle between them. Since $\overline{OP}$ and $\overline{OQ}$ are perpendicular, their images under rotation by $45°$ will also be perpendicular.

   b. Consider points $R(-3,6)$, $S(7,8)$, and $T(-4,11)$. Are $\overline{RS}$ and $\overline{RT}$ perpendicular? Explain without using triangles or the Pythagorean theorem.

   Points R, S, and T are images of the original points O, P, and Q after translating left by 3 units and up by 6 units. Since $\overline{OP}$ and $\overline{OQ}$ are perpendicular, their images under the same translation will also be perpendicular.

4. A robot that picks up tennis balls is on a straight path from $P(5,4)$ toward a ball located at $Q(-2,0)$ and then turns $90°$ to the left. Which of the following balls lies in its path?
   a. A ball located at $(5,-4)$
   b. A ball located at $(4,-5)$
   c. A ball located at $(7,-2)$
   d. A ball located at $(2,-7)$

Slope of $\overleftrightarrow{PQ}$: $\frac{4-0}{5-(-2)} = \frac{4}{7}$

> Any line perpendicular to $\overleftrightarrow{PQ}$ will have slope $-\frac{7}{4}$, so I will compute the slope of the four possible lines.

Slope of the line through Q and $(5,-4)$: $\frac{0-(-4)}{-2-5} = \frac{4}{-7} = -\frac{4}{7}$

Slope of the line through Q and $(4,-5)$: $\frac{0-(-5)}{-2-4} = \frac{5}{-6} = -\frac{5}{6}$

Slope of the line through Q and $(7,-2)$: $\frac{0-(-2)}{-2-7} = \frac{2}{-9} = -\frac{2}{9}$

Slope of the line through Q and $(2,-7)$: $\frac{0-(-7)}{-2-2} = \frac{7}{-4} = -\frac{7}{4}$

> If I call the location of the ball that is in the robot's path R, then I need to figure out which point satisfies $\overline{QP} \perp \overline{QR}$. I can use slopes to figure out which is the right point R.

The ball located at $(2,-7)$ lies in the path of the robot.

5. Gretchen thinks that the line through the points $A(-3,2)$ and $B(11,-5)$ is perpendicular to a line with slope 2. Do you agree with her?

The slope of $\overleftrightarrow{AB}$ is $\frac{-5-2}{11-(-3)}$, which is $-\frac{1}{2}$.
Any line with slope 2 is perpendicular to $\overleftrightarrow{AB}$, so Gretchen is correct.

> I remember that two lines are perpendicular if their slopes are negative reciprocals; that is, if $m_1 = -\frac{1}{m_2}$, where m_1 and m_2 are the slopes of the two lines.

Lesson 6: Segments That Meet at Right Angles

1. Two segments, $\overline{OA}$ and $\overline{OB}$, intersect at the origin O. Given the points $A(1,-4)$ and $B(-4,-3)$, is $\overline{OA} \perp \overline{OB}$? Explain.

If the two segments are perpendicular and intersect at the origin, then it must be true that $x_1 \cdot x_2 + y_1 \cdot y_2 = 0$.

$$(1) \cdot (-4) + (-4)(-3) = 0$$
$$-4 + 12 = 0 \quad \textit{False}$$

Since $8 \neq 0$, points A and B do not satisfy the criterion for perpendicularity, so $\overline{OA}$ is not perpendicular to $\overline{OB}$.

> I think that perpendicular lines have slopes that are negative reciprocals, but I haven't proven it to be true in all cases, so I cannot use it in my argument. I will have to rely on the criterion for perpendicularity.

2. Two segments, $\overline{MN}$ and $\overline{MP}$, intersect at the point $M(-1,-1)$. Given the points $N(1,-4)$ and $P(-4,-3)$, is $\overline{MN} \perp \overline{MP}$? Explain.

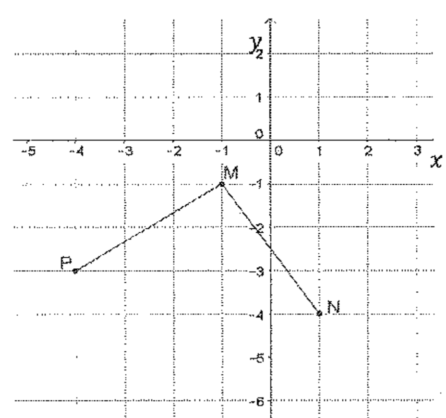
A translation along the vector $\langle 1, 1 \rangle$ maps the intersection point M to the origin. The same translation maps $P \to P'(-4+1, -3+1) = P'(-3,-2)$, and maps $N \to N'(1+1, -4+1) = N'(2,-3)$.

If $\overline{M'N'} \perp \overline{M'P'}$, then $x_1 \cdot x_2 + y_1 \cdot y_2 = 0$ where $N'(x_1, y_1)$ and $P'(x_2, y_2)$.

$$(2) \cdot (-3) + (-3) \cdot (-2) = 0$$
$$-6 + 6 = 0 \quad \textit{True}$$

$\overline{M'N'} \perp \overline{M'P'}$ by the criterion for perpendicularity, and since translations preserve angle measure by mapping lines to parallel lines, it must be true that $\overline{MN} \perp \overline{MP}$.

> The points N and P have the same coordinates as A and B above in Problem 1, but the segments do not intersect at the origin. If a translation of the plane mapping M to the origin shows that the images of the segments are perpendicular, then I can conclude that the given segments are perpendicular since translations preserve distance and angle measure.



3. Given the points $F(-2,4)$ and $H(4,6)$, find a point G where $x = 2$ such that $\overline{FG} \perp \overline{GH}$.

Let G have the coordinate $(2, g)$. Then a translation mapping G to the origin would be along the vector $\langle -2, -g \rangle$ and would map $F \to F'(-2-2, 4-g) = F'(-4, 4-g)$ and $H \to H'(4-2, 6-g) = H'(2, 6-g)$.

By the criterion for perpendicularity,

$$(-4) \cdot (2) + (4-g) \cdot (6-g) = 0$$
$$(-8) + (24 - 10g + g^2) = 0$$
$$g^2 - 10g + 16 = 0$$
$$(g-8)(g-2) = 0$$
$$g = 8 \text{ or } g = 2$$

This quadratic equation means that I am going to have two possible answers to this problem. I will have to use the *zero product property* to find the solutions for g.

The coordinates of point G whose x-coordinate is 2 such that $\overline{FG} \perp \overline{GH}$ are $G(2,8)$ or $G(2,2)$.

Lesson 7: Equations for Lines Using Normal Segments

1. Describe what a *normal* segment is.

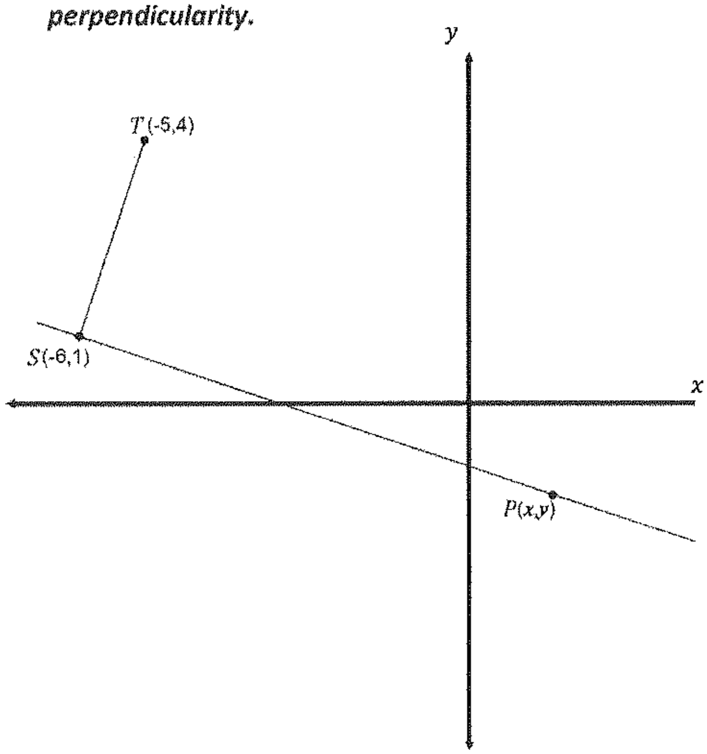
   A normal segment is a segment with one endpoint on a line to which it is perpendicular.

2. A line passes through point S of segment ST and a point $P(x, y)$.

   a. Describe what must be true if $\overline{ST} \perp \overleftrightarrow{SP}$.

   For $\overline{ST} \perp \overleftrightarrow{SP}$, S must be translated to the origin, and the coordinates of the translated coordinates of T and P by $(6, -1)$ would satisfy the criterion of perpendicularity.

I must remember that the original criterion for perpendicularity is based on two segments, each with an endpoint at the origin. If the lines in question are not passing through the origin, we can use a modified version of the criterion for perpendicularity based on translating the lines so that they pass through the origin; see Lesson 6 for reference.



b. If $\overline{ST} \perp \overleftrightarrow{SP}$, what is the equation of $\overleftrightarrow{SP}$?

$S'(0,0)$

$T'(1,3)$

$P'(x-(-6), y-1)$

$$(-5-(-6))(x-(-6))+(4-1)(y-1)=0$$
$$(1)(x+6)+(3)(y-1)=0$$
$$3(y-1)=-x-6$$
$$3y-3=-x-6$$
$$3y=-x-3$$
$$y=-\frac{1}{3}x-1$$

3. Given $M(6,10)$ and $N(-4,5)$.

A line passes through M and is perpendicular to $\overline{MN}$. Using what you know about the abscissa and ordinate values of the image point of translated point N', write the equation of the line in standard form.

If the equation of the line that passes through M is written in standard form $Ax+By=C$, then the values of A and B can be determined by the x- and y-coordinates (abscissa and ordinate coordinates, respectively) of the translated endpoint of the perpendicular segment that is not on the line, or in this case, N'.

$N'(-4-6, 5-10)$, *or* $N'(-10,-5)$

A point P on the line that passes through M becomes $P'(x-6, y-10)$.

Then, the equation of the line that passes through M and is perpendicular to $\overline{MN}$ in standard form is

$$(-10)(x-6)+(-5)(y-10)=0.$$

Lesson 8: Parallel and Perpendicular Lines

1. Given a point $A(3, 5)$ and a line $y = \frac{1}{2}x - 5$.
   a. What is the slope of any line parallel to the given line? Explain.

      The slope is $\frac{1}{2}$; lines are parallel if and only if they have equal slopes.

   b. What is the slope of any line perpendicular to the given line? Explain.

      The slope is -2; perpendicular lines have slopes that are negative reciprocals of each other.

   c. Find the equation of the line through A and parallel to the line.

      $$(y - 5) = \frac{1}{2}(x - 3)$$
      $$y = \frac{1}{2}x + 3.5$$

   d. Find the equation of the line through A and perpendicular to the line.

      $$(y - 5) = -2(x - 3)$$
      $$y = -2x + 11$$

2. Write the equation of the line through $(4, -2)$ and:
   a. Parallel to $y = -\frac{2}{3}x - 7$.

      $$y = -\frac{2}{3}x + \frac{2}{3}$$

   b. Perpendicular to $y = \frac{4}{5}x + 2$.

      $$y = -\frac{5}{4}x + 3$$

3. $\overline{AB}$ has endpoints $A(a_1, a_2)$ and $B(b_1, b_2)$, and $\overline{CD}$ has endpoints $C(c_1, c_2)$ and $D(d_1, d_2)$.

a. What is the slope of $\overline{AB}$?

$$m_{AB} = \frac{b_2 - a_2}{b_1 - a_1}$$

b. What is the slope of $\overline{CD}$?

$$m_{CD} = \frac{d_2 - c_2}{d_1 - c_1}$$

c. $\overline{AB} \perp \overline{CD}$. What is known about the relationship between the slopes of the segments?

The product of the slopes of the perpendicular segments is -1*;* $m_{AB}m_{CD} = -1$.

d. Use your responses to parts (a) and (b) to rewrite the relationship between the slopes of the segments?

$$m_{AB}m_{CD} = -1$$

$$\frac{b_2 - a_2}{b_1 - a_1} \cdot \frac{d_2 - c_2}{d_1 - c_1} = -1$$

e. What is the criterion for perpendicularity?

$$(b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2) = 0$$

I must remember that given points $O(0,0)$, $A(a_1, a_2)$, $B(b_2, b_2)$, $C(c_1, c_2)$, and $D(d_1, d_2)$, $\overline{OA} \perp \overline{OB}$ if and only if $(a_1)(b_1) + (a_2)(b_2) = 0$. $\overline{AB} \perp \overline{CD}$ if and only if $(b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2) = 0$.

f. Use algebraic manipulation to show how the relationship between the perpendicular slopes (part (d)) is related to the criterion for perpendicularity (part (e)).

$$\frac{b_2 - a_2}{b_1 - a_1} \cdot \frac{d_2 - c_2}{d_1 - c_1} = -1$$

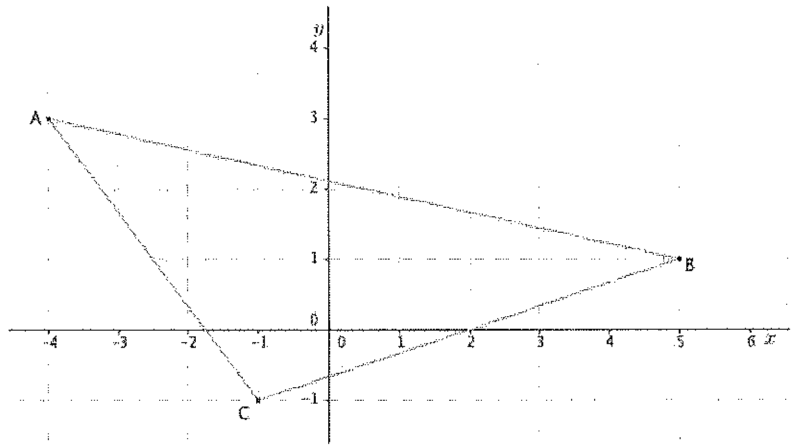
$$\frac{b_2 - a_2}{b_1 - a_1} = -\left(\frac{d_1 - c_1}{d_2 - c_2}\right)$$

$$(b_2 - a_2)(d_2 - c_2) = -(b_1 - a_1)(d_1 - c_1)$$

$$(b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2) = 0$$

Lesson 9: Perimeter and Area of Triangles in the Cartesian Plane

Using the figure below, find:



a. The perimeter of $\triangle ABC$.

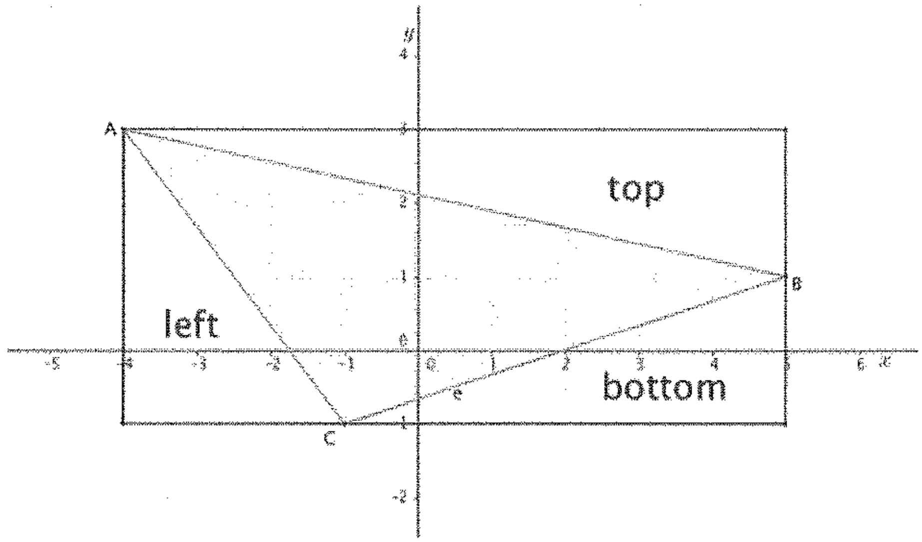
$AB = \sqrt{85}, BC = 2\sqrt{10}, CA = 5$

$\text{Perimeter}(\triangle ABC) = \sqrt{85} + 2\sqrt{10} + 5$

$\text{Perimeter}(\triangle ABC) \approx 20.5$

The perimeter of $\triangle ABC$ *is approximately* 0.5 *units.*

b. The area of $\triangle ABC$, using the decomposition method



$\mathbf{Area(left)} = \frac{1}{2}(3)(4) = 6$

$\mathbf{Area(top)} = \frac{1}{2}(2)(9) = 9$

$\mathbf{Area(bottom)} = \frac{1}{2}(2)(6) = 6$

$\mathbf{Area(rectangle)} = (4)(9) = 36$

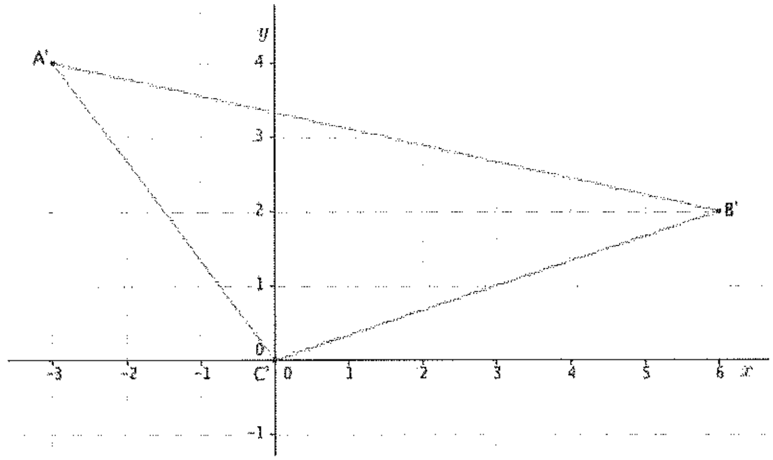
$\mathbf{Area}(\triangle ABC) = 36 - (6 + 9 + 6)$

$\mathbf{Area}(\triangle ABC) = 15$

The area of $\triangle ABC$ *is* 15 *units*^{2}.

c. The area of $\triangle ABC$, after translating it so that one vertex is at the origin and using the formula Area $= \frac{1}{2}(x_1y_2 - x_2y_1)$

$\triangle A'B'C'$ is the result of $\triangle ABC$ translated by $\langle 1, 1\rangle$ and is congruent to $\triangle ABC$.



B' is assigned (x_1, x_2); A' is assigned (y_1, y_2). Then $x_1 = 6$, $y_1 = 2$, $x_2 = -3$, and $y_2 = 4$.

$$\text{Area}(\triangle A'B'C') = \frac{1}{2}(x_1y_2 - x_2y_1)$$

$$\text{Area}(\triangle A'B'C') = \frac{1}{2}[(6 \cdot 4) - (-3 \cdot 2)]$$

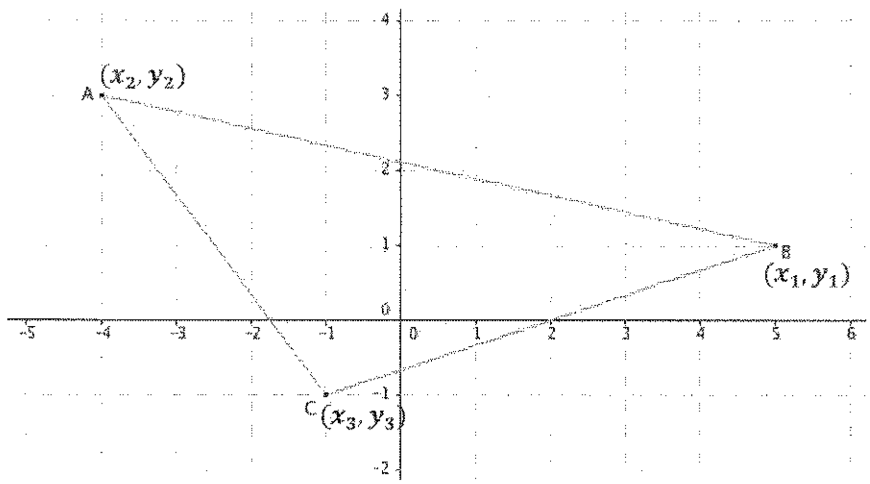
$$\text{Area}(\triangle A'B'C') = \frac{1}{2}(30)$$

$$\text{Area}(\triangle A'B'C') = 15$$

I should remember that this formula depends on the coordinate (x_1, y_1) belonging to the vertex that lies in the counterclockwise direction from the vertex at the origin.

The area of $\triangle A'B'C'$ is 15 units^2.

d. The area of $\triangle ABC$, using the shoelace formula



$\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1)$

$x_1 = 5, y_1 = 1$

$x_2 = -4, y_2 = 3$

$x_3 = -1, y_3 = -1$

I should remember that the shoelace formula is a generalization of the area formula used in part (c); when using the shoelace formula, we do not translate the triangle so that one vertex is at the origin. It is important to select vertices in the counterclockwise direction so that area calculated is positive.

$\text{Area}(\triangle ABC) = \frac{1}{2}\big((5)(3) + (-4)(-1) + (-1)(1) - (1)(-4) - (3)(-1) - (-1)(5)\big)$

$\text{Area}(\triangle ABC) = \frac{1}{2}\big(15 + 4 + (-1) - (-4) - (-3) - (-5)\big)$

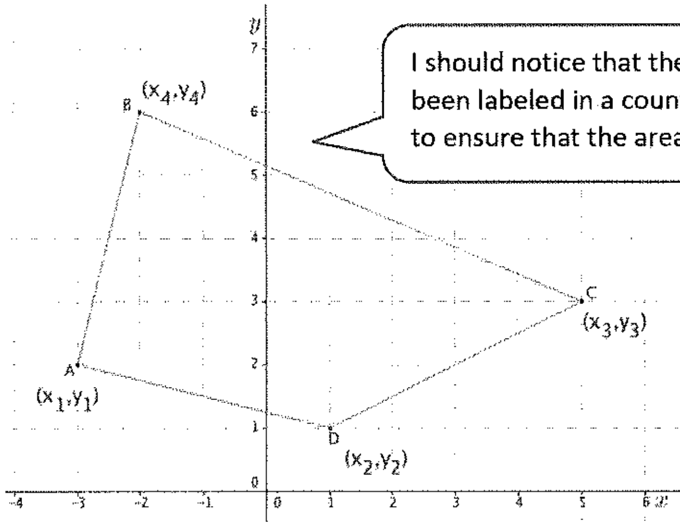
$\text{Area}(\triangle ABC) = \frac{1}{2}(30)$

$\text{Area}(\triangle ABC) = 15$

The area of $\triangle ABC$ *is* 15 *units*^{2}.

Lesson 10: Perimeter and Area of Polygonal Regions in the Cartesian Plan

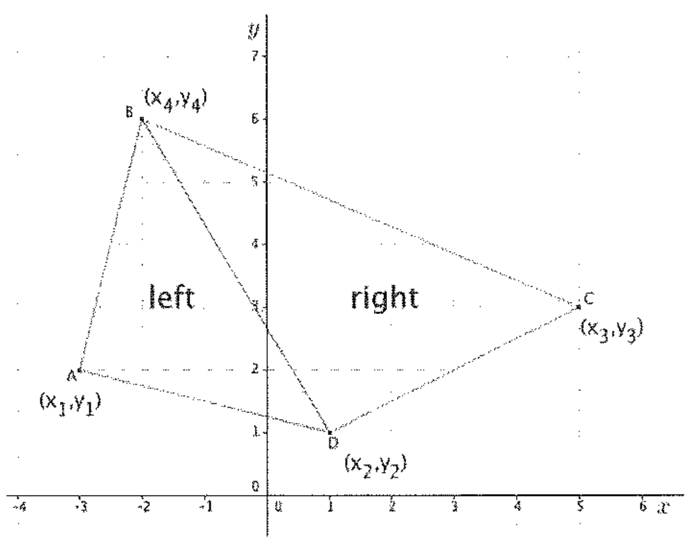
Quadrilateral $ABCD$ has the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4).



I should notice that the coordinates have been labeled in a counterclockwise direction to ensure that the area is positive.

a. Decompose the quadrilateral into triangles to find the area of the entire quadrilateral. What are the coordinates of your first triangle? What are the coordinates of the second triangle? Show the calculation for the area of the entire quadrilateral.

Possible decomposition:



Coordinates of left triangle: $(x_1, y_1), (x_2, y_2), (x_4, y_4)$

Coordinates of right triangle: $(x_2, y_2), (x_3, y_3), (x_4, y_4)$

I should notice that in taking the sum of two triangles' areas, the terms that involve the coordinates of the endpoints of the segment that decomposes the quadrilateral into the two triangles are the terms that cancel.

$$\text{Area(left)} = \frac{1}{2}(x_1y_2 + x_2y_4 + x_4y_1 - y_1x_2 - y_2x_4 - y_4x_1)$$

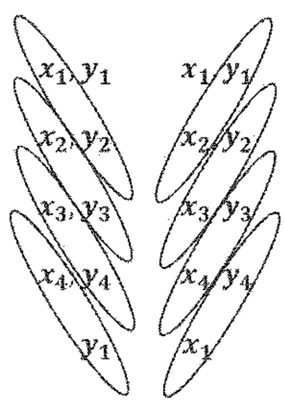
$$\text{Area(right)} = \frac{1}{2}(x_2y_3 + x_3y_4 + x_4y_2 - y_2x_3 - y_3x_4 - y_4x_2)$$

$$\text{Area(quadrilateral)} = \text{Area(left)} + \text{Area(right)}$$

$$\text{Area(quadrilateral)} = \frac{1}{2}(x_1y_2 + \cancel{x_2y_4} + x_4y_1 - y_1x_2 - \cancel{y_2x_4} - y_4x_1) + \frac{1}{2}(x_2y_3 + x_3y_4 + \cancel{x_4y_2} - y_2x_3 - y_3x_4 - \cancel{y_4x_2})$$

$$\text{Area(quadrilateral)} = \frac{1}{2}(x_1y_2 + x_4y_1 + x_2y_3 + x_3y_4 - y_1x_2 - y_4x_1 - y_2x_3 - y_3x_4)$$

b. Use your answer from part (a) to show how the formula is indeed a "shoelace" formula.



Just as with the shoelace formula for the area of a triangle, the arrangement of terms is presented such that the first column is the sum of terms, while the second column is the set of terms to be subtracted.

c. Identify the numeric coordinates of the quadrilateral, and find its area.

$A(-3,2)$ x_1, y_1

$D(1,1)$ x_2, y_2

$C(5,3)$ x_3, y_3

$B(-2,6)$ x_4, y_4

$$\text{Area(quadrilateral)} = \frac{1}{2}(x_1y_2 + x_4y_1 + x_2y_3 + x_3y_4 - y_1x_2 - y_4x_1 - y_2x_3 - y_3x_4)$$

$$\text{Area(quadrilateral)} = \frac{1}{2}\big((-3)(1) + (-2)(2) + (1)(3) + (5)(6) - (2)(1) - (6)(-3) - (1)(5) - (3)(-2)\big)$$

$$\text{Area(quadrilateral)} = \frac{1}{2}\big((-3) + (-4) + (3) + (30) - (2) - (-18) - (5) - (-6)\big)$$

$$\text{Area(quadrilateral)} = 21.5$$

Lesson 11: Perimeters and Areas of Polygonal Regions Defined by Systems of Inequalities

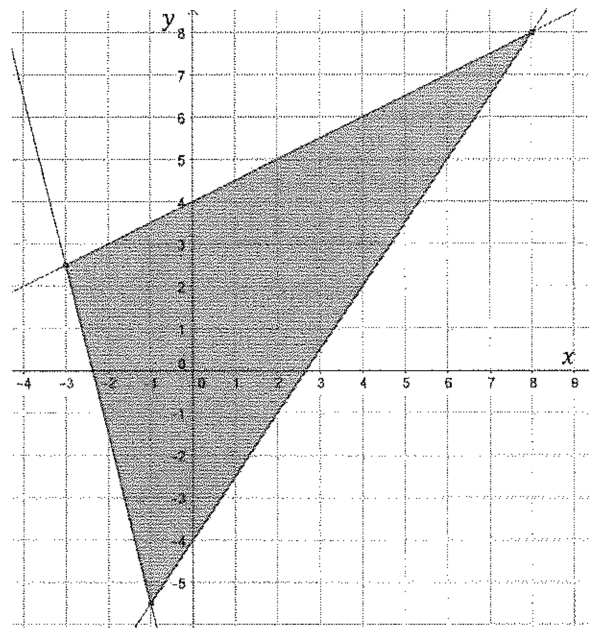
Given the system of inequalities below:

$y \le \frac{1}{2}x + 4 \qquad y \ge \frac{3}{2}x - 4 \qquad y \ge -4x - 9.5$

a. Graph the region defined by the system.

See the diagram to the right.

Graphing a system of inequalities is just like graphing a system of equations. I can start by graphing the lines representing the corresponding equations $y = \frac{1}{2}x + 4$, $y = \frac{3}{2}x - 4$, and $y = -4x - 9.5$. Then I just have to figure out which half-plane represents the solution set to each inequality.



b. Identify the vertices of the region.

The vertices of the region are the intersections of the graphs that define the region's figure.

$y = \frac{1}{2}x + 4 \qquad y = \frac{3}{2}x - 4 \qquad y = -4x - 9.5$

Using substitution with the above equations:

$$\frac{1}{2}x + 4 = \frac{3}{2}x - 4$$
$$-x = -8$$
$$x = 8$$
$$y = \frac{1}{2}(8) + 4$$
$$y = 8$$
$$(8, 8)$$

$$\frac{1}{2}x + 4 = -4x - 9.5$$
$$\frac{9}{2}x = -13.5$$
$$x = -3$$
$$y = \frac{1}{2}(-3) + 4$$
$$y = 2.5$$
$$(-3, 2.5)$$

$$\frac{3}{2}x - 4 = -4x - 9.5$$
$$\frac{11}{2}x = -5.5$$
$$x = -1$$
$$y = -4(-1) - 9.5$$
$$y = -5.5$$
$$(-1, -5.5)$$

The vertices of the region defined by the inequalities are $(8, 8)$, $(-3, 2.5)$, *and* $(-1, -5.5)$.

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c. Find the perimeter of the region defined by the system of inequalities. Round your answer to the nearest tenth of a unit if necessary.

I can calculate the perimeter by finding the length of the three sides. To find their lengths, I can use the distance formula.

Using the distance formula:

Let d_1 *be the distance between* $(8, 8)$ *and* $(-3, 2.5)$.

$d_1 = \sqrt{(-3-8)^2 + (2.5-8)^2}$

$d_1 = \sqrt{121 + 30.25}$

$d_1 \approx 12.3$

Let d_2 *be the distance between* $(-3, 2.5)$ *and* $(-1, -5.5)$.

$d_2 = \sqrt{(-1-(-3))^2 + (-5.5-2.5)^2}$

$d_2 = \sqrt{4 + 64}$

$d_2 \approx 8.2$

Let d_3 *be the distance between* $(-1, -5.5)$ *and* $(8, 8)$.

$d_3 = \sqrt{(-1-8)^2 + (-5.5-8)^2}$

$d_3 = \sqrt{81 + 182.25}$

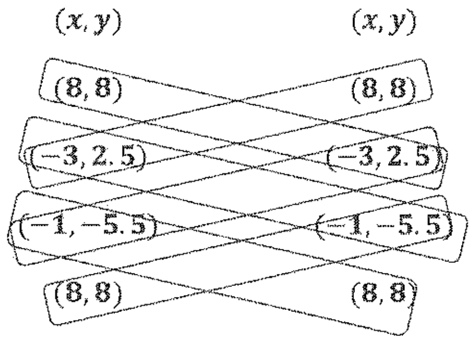
$d_3 \approx 16.2$

$\text{Perimeter} = d_1 + d_2 + d_3$

$\text{Perimeter} \approx 12.3 + 8.2 + 16.2 \approx 36.7$

The perimeter of the region is approximately 36.7 *units.*

d. Using Green's theorem, use the vertices found in part (b) to calculate the area of the region graphed in part (a).



Green's theorem is a lot easier to organize using the *shoelace* approach. That will help me determine which products to add and which products to subtract.

By Green's theorem, the area of the figure can be calculated:

$$\text{Area} = \frac{1}{2}\big((8 \cdot 2.5) + (-3 \cdot -5.5) + (-1 \cdot 8) - (8 \cdot -3) - (2.5 \cdot -1) - (-5.5 \cdot 8)\big)$$

$$\text{Area} = \frac{1}{2}\big((20) + (16.5) + (-8) - (-24) - (-2.5) - (-44)\big)$$

$$\text{Area} = \frac{1}{2}(20 + 16.5 - 8 + 24 + 2.5 + 44)$$

$$\text{Area} = \frac{1}{2}(99)$$

$$\text{Area} = 49.5$$

The area of the region defined by the system of inequalities is 49.5 ***square units.***

e. Calculate the area of the region in a different way to confirm your answer to part (c).

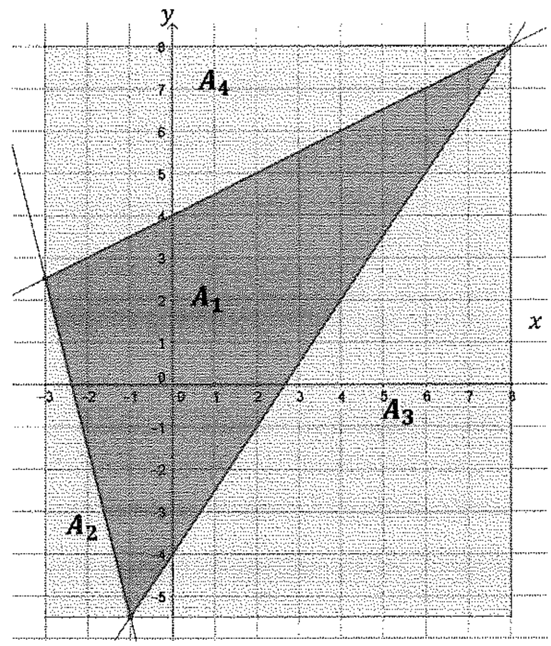
The region can be composed into a rectangular region. Using properties of area, the area of the region defined by the system of inequalities, A_1, is equal to the area of the rectangle, A_T, minus the areas of the outer right triangles, A_2, A_3, and A_4.

Since I know the vertices of the region, I can compose the triangle into a larger rectangle and remove the areas of the outer triangles to reveal the remaining area of the triangular region defined by the system.

$$A_T = A_1 + A_2 + A_3 + A_4$$

$$A_T = 11 \cdot 13.5$$

$$A_T = 148.5$$



$$A_2 = \frac{1}{2}(2 \cdot 8)$$

$$A_2 = 8$$

$$A_3 = \frac{1}{2}(9 \cdot 13.5)$$

$$A_3 = 60.75$$

$$A_4 = \frac{1}{2}(5.5 \cdot 11)$$

$$A_4 = 30.25$$

$$148.5 = A_1 + 8 + 60.75 + 30.25$$

$$148.5 = A_1 + 99$$

$$148.5 - 99 = A_1$$

$$49.5 = A_1$$

Using decomposition, the area of the region defined by the system of inequalities is confirmed to be 49.5 square units.

Lesson 12: Dividing Segments Proportionately

1. Given points $A(-8,3)$ and $B(2,-3)$, find point C on $\overline{AB}$ such that the ratio of AC to CB is $2:1$.

$C\left(-8+\frac{2}{3}(2-(-8)), 3+\frac{2}{3}(-3-3)\right)$

$C\left(-8+\frac{2}{3}(10), 3+\frac{2}{3}(-6)\right)$

$C\left(-8+\frac{20}{3}, 3+(-4)\right)$

$C\left(-\frac{24}{3}+\frac{20}{3}, -1\right)$

$C\left(-\frac{4}{3}, -1\right)$

The point C that cuts $\overline{AB}$ such that the ratio of AC to CB is $2:1$ is $\left(-\frac{4}{3}, -1\right)$.

> If $\overline{AB}$ is decomposed into two parts in the ratio $2:1$, then the associated part to whole ratio of AC to AB must be $2:3$. This means that C must be $\frac{2}{3}$ of the distance from A to B.

> If C is $\frac{2}{3}$ of the distance from A to B, then I can start with the x-coordinate of A and add $\frac{2}{3}$ the horizontal distance between A and B. Then I can use the y-coordinate of A and add $\frac{2}{3}$ the vertical distance between A and B.

2. Given points $U(2,-4)$, $V(10,5)$, and $W(x,y)$ such that V lies on $\overline{UW}$, locate W if $UV = \frac{5}{4}VW$.

If $UV = \frac{5}{4}VW$, then the part to part ratio of UV to VW is $5:4$. The associated part to whole ratio of UV to UW is then $5:9$, so point V lies $\frac{5}{9}$ of the distance from U to W.

$V(10,5) = \left(2+\frac{5}{9}(x-2), -4+\frac{5}{9}(y-(-4))\right)$

$$10 = 2+\frac{5}{9}(x-2) \qquad 5 = -4+\frac{5}{9}(y-(-4))$$

$$8 = \frac{5}{9}(x-2) \qquad 9 = \frac{5}{9}(y+4)$$

$$\frac{72}{5} = x-2 \qquad \frac{81}{5} = y+4$$

$$\frac{72}{5}+2 = x \qquad \frac{81}{5}-4 = y$$

$$\frac{82}{5} = x \qquad \frac{61}{5} = y$$

> The coordinates of V are both greater than the coordinates of U. If V lies on $\overline{UW}$ then I know that the coordinates of W must be greater than both the coordinates of U and of V.

> The ordered pairs representing the location of V give me equivalent expressions. I can use equations to solve for the unknown values x and y.

The coordinates of point W are $\left(\frac{82}{5}, \frac{61}{5}\right)$, or $(16.4, 12.2)$.

3. A robot starts at the point $(0, 2)$ and travels along a straight line toward $(6, 7)$ at a constant speed. The robot passes $(6, 7)$ at exactly 10 minutes after it started.

   a. What is the robot's approximate speed in units per minute?

   $d = \sqrt{(6-0)^2 + (7-2)^2}$

   $d = \sqrt{6^2 + 5^2}$

   $d = \sqrt{36 + 25}$

   $d = \sqrt{61}$

   $d \approx 7.8$

   Once I know the distance from $(0, 2)$ to $(6, 7)$, I can use $d = rt$ to find the robot's speed.

   The robot travels a distance of approximately 7.8 *units.*

   $d = r \cdot t$

   $\sqrt{61} = r \cdot 10$

   $\frac{\sqrt{61}}{10} = r$

   $0.78 \approx r$

   The robot's speed is approximately 0.78 *units per minute.*

   b. If the robot continues at the same constant rate and direction, where will it be at 1 hour?

   $1 \text{ hr} = 60 \text{ min} = 6 \cdot (10 \text{ min})$

   The distance to the point where the robot lies at 1 *hour will be six times the horizontal and vertical distances from* $(0, 2)$ *to* $(6, 7)$.

   $\left(0 + 6(6-0), 2 + 6(7-2)\right)$

   $(0 + 36, 2 + 30)$

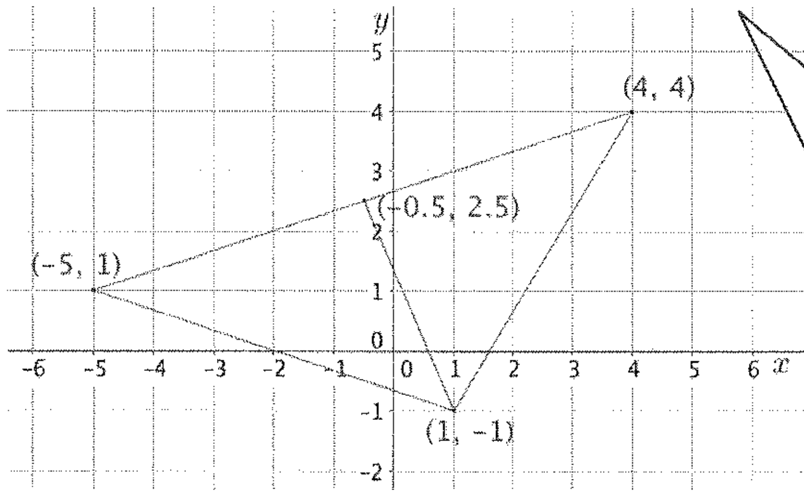
   $(36, 32)$

   At 1 *hour, the robot will be at the point* $(36, 32)$.

   Once I know the distance from $(0, 2)$ to $(6, 7)$, I can use $d = rt$ to find the robot's speed. If its speed is constant, then its horizontal and vertical distance at time 60 minutes will be 6 times what is was at 10 minutes.

Lesson 13: Analytic Proofs of Theorems Previously Proved by Synthetic Means

1. Draw a median from one of the three vertices of the triangle shown on the grid below.



> I know that a median is a segment drawn in a triangle that joins a vertex of the triangle with the midpoint of the opposite side. To draw a median, I have to find the midpoint of one of the sides of the triangle. To find the midpoint, I can use the midpoint formula.

Sample solution: midpoint joining $(-5, 1)$ ***and*** $(4, 4)$:

$$\left(\frac{1}{2}(4 + (-5)), \frac{1}{2}(4 + 1)\right)$$

$$\left(\frac{1}{2}(-1), \frac{1}{2}(5)\right)$$

$$\left(-\frac{1}{2}, \frac{5}{2}\right)$$

The midpoint of the side joining $(-5, 1)$ ***and*** $(4, 4)$ ***is*** $\left(-\frac{1}{2}, \frac{5}{2}\right)$.

a. Find the point on the median that is $\frac{2}{3}$ the distance from the vertex to the midpoint of the opposite side.

$$\left(1+\frac{2}{3}\left(-\frac{1}{2}-1\right), -1+\frac{2}{3}\left(\frac{5}{2}-(-1)\right)\right)$$

$$\left(1+\frac{2}{3}\left(-\frac{3}{2}\right), -1+\frac{2}{3}\left(\frac{7}{2}\right)\right)$$

$$\left(1+(-1), -1+\frac{7}{3}\right)$$

$$\left(0, \frac{4}{3}\right)$$

I can use the formula I learned from the last lesson using the point $(1,-1)$ as a starting point.

The point on the median $\frac{2}{3}$ the distance from the vertex to its opposite endpoint is $\left(0, \frac{4}{3}\right)$.

b. If you were to follow the above directions for a different median of the triangle, what would your results be and why?

The results should be the same no matter which vertex the median is drawn from. In the Opening Exercise from this lesson, it was shown that the medians of a triangle meet at a point that is $\frac{1}{3}$ the distance from the midpoint of a side to the opposite vertex. This problem requires finding the point $\frac{2}{3}$ the distance from a vertex to the midpoint of the opposite side, which happens to be the same point.

If I find a point on a segment that is $\frac{2}{3}$ the length of the segment from one endpoint, then it must be $\frac{1}{3}$ the length of the segment from the other endpoint.

2. Given the quadrilateral with vertices $(-5,1)$, $(-1,4)$, $(3,1)$, and $(-1,-2)$.

a. Is it a trapezoid?

I need to calculate the slopes of a pair of opposite sides.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_1 = \frac{4-1}{-1-(-5)} \qquad m_2 = \frac{1-(-2)}{3-(-1)}$$

$$m_1 = \frac{3}{4} \qquad m_2 = \frac{3}{4}$$

The slopes of one pair of opposite sides are equal, so they are parallel. If a quadrilateral has at least one pair of opposite parallel sides, then the quadrilateral is a trapezoid.

b. Is it a parallelogram?

I need to calculate the slopes of the other pair of opposite sides.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$m_3 = \frac{1-(-2)}{-5-(-1)}$ $m_4 = \frac{4-1}{-1-3}$

$m_3 = -\frac{3}{4}$ $m_4 = -\frac{3}{4}$

The slopes of the second pair of opposite sides are also equal, so both pairs of opposite sides are parallel. If a quadrilateral has both pairs of opposite sides parallel, then it is a parallelogram.

c. Is it a rectangle?

I need to compare the slopes of one pair of consecutive sides.

The slopes of consecutive sides are $\frac{3}{4}$ and $-\frac{3}{4}$, which are opposites, but are not opposite reciprocals (or negative reciprocals). This means that consecutive sides are not perpendicular, so this is not a rectangle.

d. Is it a rhombus?

I've already shown that the quadrilateral is a parallelogram, so to be a rhombus, I can show that consecutive sides are the same length or that the diagonals are perpendicular.

The diagonals of the quadrilateral lie on the same horizontal and vertical lines, and since horizontal and vertical lines are perpendicular, the diagonals of the quadrilateral are perpendicular. A parallelogram with perpendicular diagonals is a rhombus.

e. Is it a square?

To be a square, a quadrilateral has to be both a rhombus and a rectangle. This already failed the rectangle test.

No, the quadrilateral is not a square because its consecutive sides are not perpendicular.

Lesson 14: Motion Along a Line—Search Robots Again

1. A robot is programmed to travel along a line segment at a constant speed. If P represents the robot's position at any given time, t, in minutes:

$P = (a_1, a_2) + \frac{t}{0.9}(b_1 - a_1, b_2 - a_2)$

> This looks just like the formula that I used to divide a line segment proportionately a few lessons ago. There is a starting point and a multiple of the horizontal and vertical distance added on.

a. Explain what (a_1, a_2) indicates in the equation.

The robot is programmed to move along a linear path, so (a_1, a_2) is the robot's position at $t = 0$.

b. Explain what $(b_1 - a_1, b_2 - a_2)$ indicates in the equation.

The total horizontal distance covered in a given time interval is represented by $b_1 - a_1$, and the total vertical distance covered under the same time interval is represented by $b_2 - a_2$.

c. Explain what $\frac{t}{0.9}$ indicates in the equation.

> The quotient $\frac{t}{0.9}$ determines the total time of travel along the line.

The 0.9 indicates that the robot is programmed to move from point (a_1, a_2) to (b_1, b_2) in a time of 0.9 minutes. The t is a factor that indicates by what multiple of the given time interval to multiply the total distance.

d. If the robot is programmed to move from $A(3, 7)$ to $B(-1, 4)$ in a timeframe of 0.9 minute, rewrite the given equation using these values.

$$P = (3, 7) + \frac{t}{0.9}(-1 - 3, 4 - 7)$$

$$P = (3, 7) + \frac{t}{0.9}(-4, -3)$$

e. What distance parallel to the horizontal axis does the robot move every 0.9 minute?

The difference in the x-coordinates of the endpoints is -4, so the robot moves 4 units horizontally in the negative direction.

f. What distance parallel to the vertical axis does the robot move every 0.9 minute?

The difference in the y-coordinates of the endpoints is -3, so the robot moves 3 units vertically in the negative direction.

g. Where would the robot be at a time of 0.3 minute?

$$P = (3,7) + \frac{t}{0.9}(-4,-3)$$

$$P = (3,7) + \frac{0.3}{0.9}(-4,-3)$$

$$P = (3,7) + \frac{1}{3}(-4,-3)$$

$$P = (3,7) + \left(-\frac{4}{3}, -\frac{3}{3}\right)$$

$$P = \left(\frac{5}{3}, 6\right)$$

At $t = 0.3$, the robot would be located at $\left(\frac{5}{3}, 6\right)$.

0.3 minute is $\frac{1}{3}$ of the time that it takes for the robot to go from endpoint to endpoint, so its location at that time will be $\frac{1}{3}$ the horizontal distance between the endpoints and $\frac{1}{3}$ the vertical distance between the endpoints.

2. Mark programs a robot to move at a constant speed along a linear path in a large warehouse. His robot starts at his location, which is the origin, and moves to point $P(215, 120)$ in a time of 4 minutes. A second robot is already in the warehouse, and a computer programs it to move at a constant speed from $C(80, 160)$ to $D(210, 20)$ in $3\frac{1}{2}$ minutes. Assume the robots begin their motion at the same time.

   a. Write an equation to model the position, P_m, of Mark's robot at any time, t, in minutes.

   $$P_m = (0,0) + \frac{t}{4}(215, 120)$$

   b. Write an equation to model the position, P_r, of the second robot at any time, t, in minutes.

   $$P_r = (80,160) + \frac{t}{3.5}(210 - 80, 20 - 160)$$

   or

   $$P_r = (80,160) + \frac{t}{3.5}(130, -140)$$

c. Do the paths of the robots intersect? If so, determine the intersection point? If not, explain why.

Path of Mark's robot

$y = \frac{120}{215}x$

or

$y = \frac{24}{43}x$

Path of second robot

$y - 160 = -\frac{140}{130}(x - 80)$

or

$y - 160 = -\frac{14}{13}(x - 80)$

I know that lines with different slopes will intersect at some point in the plane, so I need to determine the slopes of the lines that represent the paths of the robots.

The slopes of the graphs representing the paths of the robots are not equal, so the paths do intersect.

Using substitution:

$$\left(\frac{24}{43}x\right) - 160 = -\frac{14}{13}(x - 80)$$
$$\frac{24}{43}x - 160 = -\frac{14}{13}x + \frac{1120}{13}$$
$$\frac{24}{43}x + \frac{14}{13}x = \frac{1120}{13} + 160$$
$$\frac{312}{559}x + \frac{602}{559}x = \frac{1120}{13} + \frac{2080}{13}$$
$$\frac{914}{559}x = \frac{3200}{13}$$
$$x = \frac{1\,788\,800}{11882}$$
$$x = \frac{68800}{457}$$
$$x \approx 150.5$$

$$y = \frac{24}{43}x$$
$$y = \frac{24}{43} \cdot \frac{68800}{457}$$
$$y = \frac{1\,651\,200}{19651}$$
$$y = \frac{38400}{457}$$
$$y \approx 84.0$$

The paths of the robots intersect at the approximate point $(150.5, 84.0)$.

d. If the robots both leave their starting points at exactly the same time, will they collide with one another?

Mark's robot

$$P_m = (0,0) + \frac{t}{4}(215,120)$$

$$(0,0) + \frac{t}{4}(215,120) = \left(\frac{68800}{457}, \frac{38400}{457}\right)$$

$$0 + \frac{t}{4}(215) = \frac{68800}{457}$$

$$\frac{t}{4} = \frac{68800}{98255}$$

$$t = 4\left(\frac{68800}{98255}\right)$$

$$t \approx 2.8$$

For the robots to collide, they have to be at the same point at the same time. I can use the point of intersection with the equations for their locations to determine what time each robot will be at the intersection point. If the times are the same, then the robots will collide.

Second robot

$$P_r = (80,160) + \frac{t}{3.5}(130,-140)$$

$$(80,160) + \frac{t}{3.5}(130,-140) = \left(\frac{68800}{457}, \frac{38400}{457}\right)$$

$$80 + \frac{t}{3.5}(130) = \frac{68800}{457}$$

$$\frac{t}{3.5} = \frac{32240}{59410}$$

$$t = 3.5\left(\frac{32240}{59410}\right)$$

$$t \approx 1.9$$

The robots will not collide with one another because they will be at the point of intersection at different times. Mark's robot reaches the intersection point at approximately 2.8 minutes from its start, and the other robot does not reach that location until approximately 1.9 minutes after starting.

Lesson 15: The Distance from a Point to a Line

1. Given the point $K(3,7)$ and line n representing the equation $y = \frac{1}{3}x - 4$.

   a. Use the distance formula derived in the lesson to find the distance from point K to line n.

> In the formula that I derived in class, (p, q) are the coordinates of the point not on the line from which we need to find the distance. Even though the point is called K in this problem, I know that the values of p and q are 3 and 7, respectively.

$$d = \sqrt{\left(\frac{p + qm - bm}{1 + m^2} - p\right)^2 + \left(m\left(\frac{p + qm - bm}{1 + m^2}\right) + b - q\right)^2}$$

$$d = \sqrt{\left(\frac{3 + 7 \cdot \left(\frac{1}{3}\right) - (-4) \cdot \left(\frac{1}{3}\right)}{1 + \left(\frac{1}{3}\right)^2} - 3\right)^2 + \left(\frac{1}{3} \cdot \left(\frac{3 + 7\left(\frac{1}{3}\right) - (-4)\left(\frac{1}{3}\right)}{1 + \left(\frac{1}{3}\right)^2}\right) + (-4) - 7\right)^2}$$

$$d = \sqrt{\left(\frac{3 + \frac{7}{3} + \frac{4}{3}}{1 + \frac{1}{9}} - 3\right)^2 + \left(\frac{1}{3} \cdot \left(\frac{3 + \frac{7}{3} + \frac{4}{3}}{1 + \frac{1}{9}}\right) - 11\right)^2}$$

$$d = \sqrt{\left(\frac{\frac{20}{3}}{\frac{10}{9}} - 3\right)^2 + \left(\frac{1}{3} \cdot \left(\frac{\frac{20}{3}}{\frac{10}{9}}\right) - 11\right)^2}$$

$$d = \sqrt{\left(\frac{20}{3} \cdot \frac{9}{10} - 3\right)^2 + \left(\frac{1}{3} \cdot \frac{20}{3} \cdot \frac{9}{10} - 11\right)^2}$$

$$d = \sqrt{(6 - 3)^2 + (2 - 11)^2}$$

$$d = \sqrt{3^2 + (-9)^2}$$

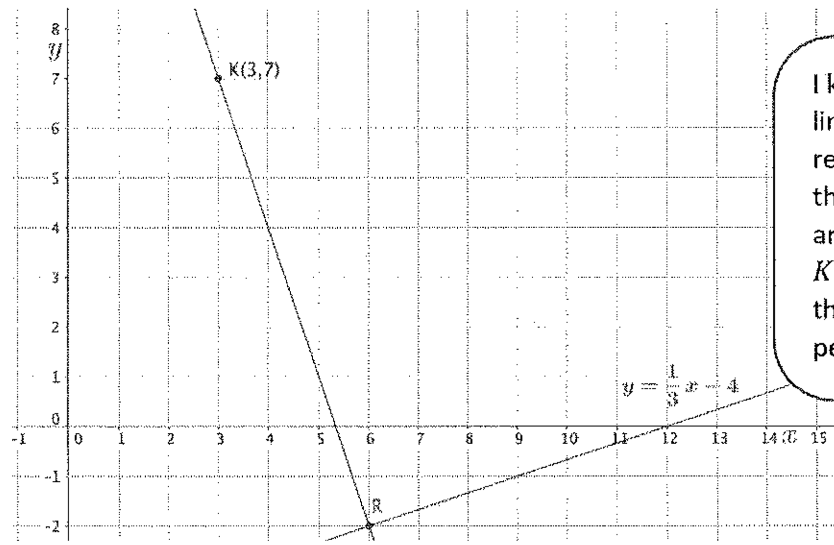
$$d = \sqrt{9 + 81}$$

$$d = \sqrt{90}$$

$$d = 3\sqrt{10}$$

Point K is $3\sqrt{10}$ units away from line n.

b. Draw the given information on a coordinate plane. Use your knowledge about the slopes of perpendicular lines and the distance formula to find the distance from point K to line n.



I know that perpendicular lines have negative reciprocal slopes. I can use the negative reciprocal slope and the coordinates of point K to write the equation of the line through K that is perpendicular to line n.

The shortest distance from K to line n is the length of $\overline{KR}$ where R is on line n and $\overline{KR} \perp n$. Perpendicular lines have negative reciprocal slopes. The slope of line n is $\frac{1}{3}$, so the slope of $\overline{KR}$ must be -3.

Using the point-slope form of the equation of a line, the equation of $\overleftrightarrow{KR}$ is $y - 7 = -3(x - 3)$. Converted to slope-intercept form, the equation is $y = -3x + 16$. The point of intersection of the lines is the solution to the system of equations: $\begin{cases} y = \frac{1}{3}x - 4 \\ y = -3x + 16 \end{cases}$.

$$\frac{1}{3}x - 4 = -3x + 16$$
$$\frac{1}{3}x + 3x = 20$$
$$\frac{10}{3}x = 20$$
$$x = 6$$

$$y = \frac{1}{3}x - 4$$
$$y = \frac{1}{3}(6) - 4$$
$$y = 2 - 4$$
$$y = -2$$

To find the point where the perpendicular lines intersect, I can solve the system of their equations. The solution that satisfies both equations will be the point of intersection. Then I can use the distance formula.

The point on line n closest to point K is $(6, -2)$.

$$d = \sqrt{(3-6)^2 + \left(7 - (-2)\right)^2}$$
$$d = \sqrt{(-3)^2 + (9)^2}$$
$$d = \sqrt{9 + 81}$$
$$d = \sqrt{90}$$
$$d = 3\sqrt{10}$$

The distance from K to line n is $3\sqrt{10}$ units.

2. Review the derivation of the distance formula for finding the distance from a point to a line. In the lesson, the coordinates of point R were rewritten in terms of p, q, m, and b using algebraic relationships. What algebraic relationships were used?

In the lesson, the coordinates of the point R, on line l, closest to point P are not given and so were first labeled (r, s). The distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, can only be directly used to find the distance PR if the coordinates of both points P and R are known. Even though the values of x_2 and y_2 are unknown, the relationship between their values is indicated by the equation of the line upon which their point lies, $y = mx + b$.

> I need the coordinates of two different points in order to calculate the distance between them using the distance formula. I only have the coordinates of one point, but I know that the coordinates of the other point are related by $y = mx + b$.

Knowing that $\overline{PR} \perp l$ provides another relationship between the x- and y-values of point P and the x- and y-values of any other point on l other than R. Choosing a point T with an x-value 1 greater than that of R is very convenient because under the translation mapping point R to R' at the origin, the coordinates of T' are $(1, m)$. The perpendicularity criterion now provides a relationship between the coordinates of P' and T', allowing us to write the value of r in terms of p, q, m, and b.

> I know the distance from the point to the line is measured perpendicularly, so the perpendicularity criterion from earlier in the module provides me with another algebraic relationship.

With the value of r now known, the corresponding y-value of r can be calculated using the equation $y = mx + b$. This provides the coordinates of two points between which the distance can be calculated using the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.