Homework Helpers

Geometry
Module 3

Lesson 1: What Is Area?

1. Describe what *area* means.

   Area is a way of associating to any region, without regard to the shape of the region, a quantity that reflects our sense of how big the region is.

2. Describe how *area* is measured.

   In order to measure area, we decide on a unit square and say that the area of a region is the number of whole and partial unit squares needed to tile the region.

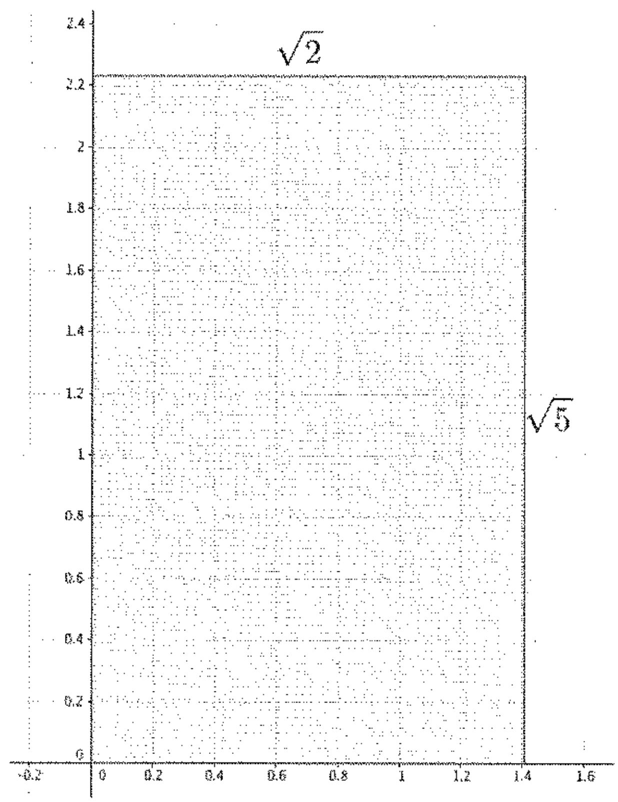
3. Why is $\text{length} \times \text{width}$ the formula used to find the area of rectangular regions?

   A rectangular region can be tiled by an appropriate unit square in a way such that the number of unit squares could be counted or determined by $\text{length} \times \text{width}$ ***since multiplication is repeated addition.***

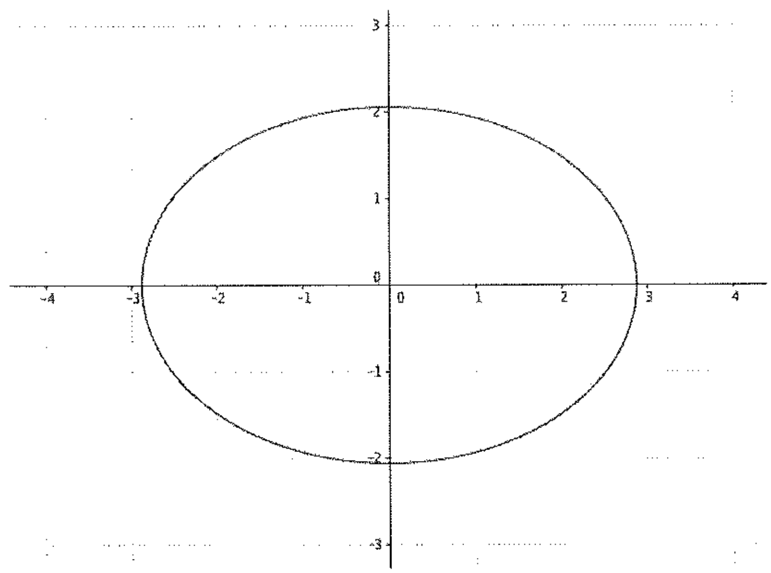
4. After observing the figure below, Sadia says that the area can be determined using the $\text{length} \times \text{width}$ formula. Libby disagrees, saying that since an irrational length has to be approximated with fractions or finite decimals, an area with irrational side lengths that are approximated to fractional or decimal lengths will only result in an approximated area. Who is right? Explain.

Sadia is right. Since we know how to find the area of rectangular regions with rational side lengths, we can examine a series of rational side-length approximations for each side: one lower approximation and one upper approximation. If each successive set of approximations is refined so that the rational values are closer to the actual irrational value, we see that the calculated value of the area is increasingly like the decimal expansion of the product of the irrational side lengths. Therefore, the area of a rectangular region with irrational side lengths can be determined by the $\text{length} \times \text{width}$ *formula because the product of the irrational side lengths is the only number that is always between the lower and upper approximations when using rational side lengths.*

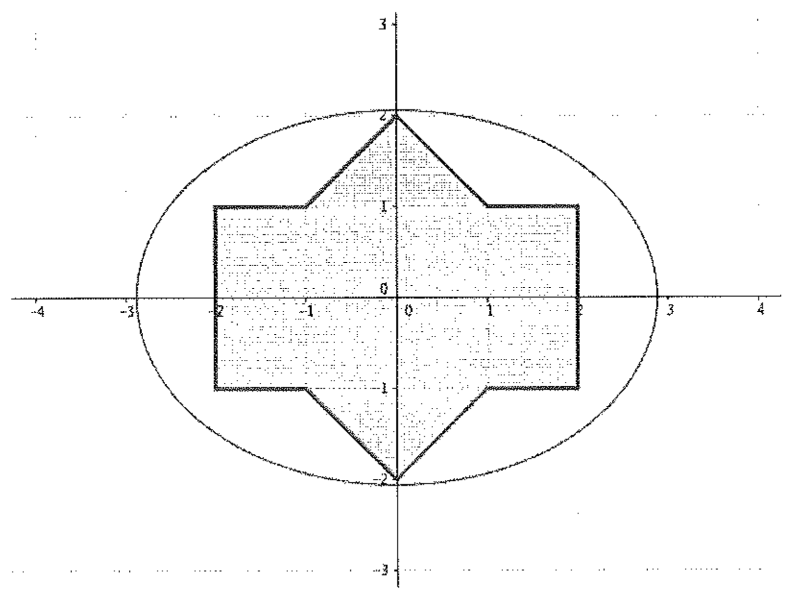
A lower approximation is determined by using the exact digits of the irrational expansion but truncated. An upper approximation is determined by using one digit greater than the last digit the truncated expansion.



5. For the following curved region, use whole and half squares to draw the polygonal regions that represent a lower approximation and upper approximation of the figure's area.

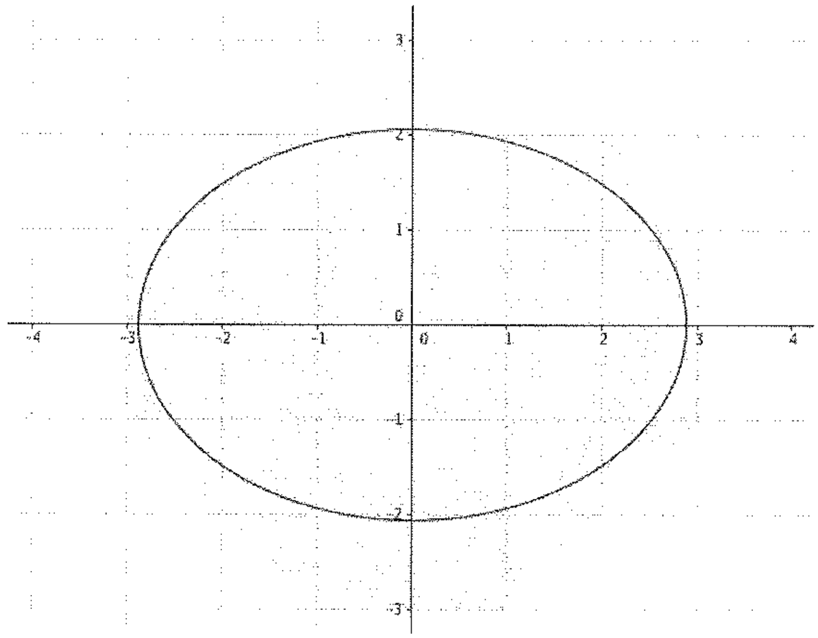


a. What is the area of the lower approximation?



10 square units

b. What is the area of the upper approximation?



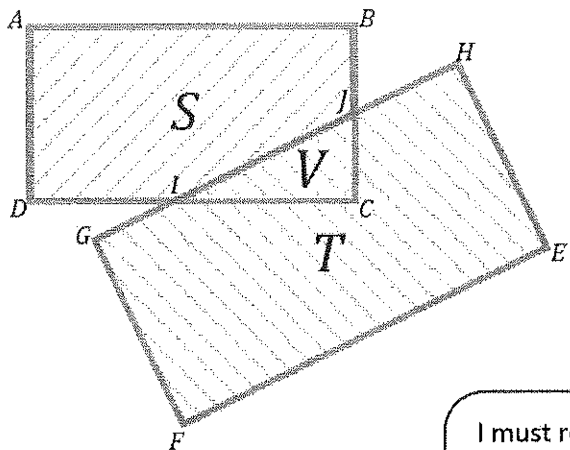
26 square units

c. How can the approximation be refined to improve the accuracy of the estimate of area?

In this instance, where we are using a grid of unit squares to approximate the area, we can use smaller squares in the grid, which will help reduce the degree by which we overestimate and underestimate.

Lesson 2: Properties of Area

1. Rectangular regions S and T are shown below. The regions overlap, forming triangular region V.



I must remember that if S and T are regions in the plane, then the union of the two regions is all the points that lie in S or T, including the points that lie in both S and T.

a. Use the appropriate notation to describe the figure $ABJHEFGID$.

$S \cup T$

b. Use the appropriate notation to describe the figure IJC.

$S \cap T$

The intersection of regions S and T is all the points that lie in both S and T.

c. In rectangle $ABCD$, $AB = 10$ and $AD = 6$. In rectangle $EFGH$, $EF = 13$ and $FG = 7$. Side $\overline{GH}$ intersects the midpoints of $\overline{CD}$ and $\overline{BC}$ at I and J, respectively. Calculate the area of figure $ABJHEFGID$.

The area of rectangle $ABCD$: $(10)(6) = 60$

The area of rectangle $EFGH$: $(13)(7) = 91$

The area of $\triangle IJC$: $\frac{1}{2}(5)(3) = 7.5$

The area of figure $ABJHEFGID$ is $60 + 91 - 7.5$, *or* 143.5.

d. Use the appropriate notation to describe your calculations in part (c).

$\text{Area}(S \cup T) = \text{Area}(S) + \text{Area}(T) - \text{Area}(S \cap T)$

2. The area of a square (or a square region) is the square of its side length. In Figure 1, a rectangle with length ℓ and width w has an area of A. Use the rectangle, the fact about the area of a square, and Figure 2 to demonstrate why the area of a rectangle must be length × width.

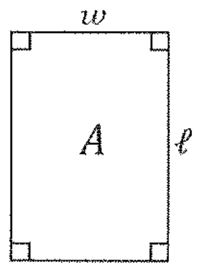


Figure 1

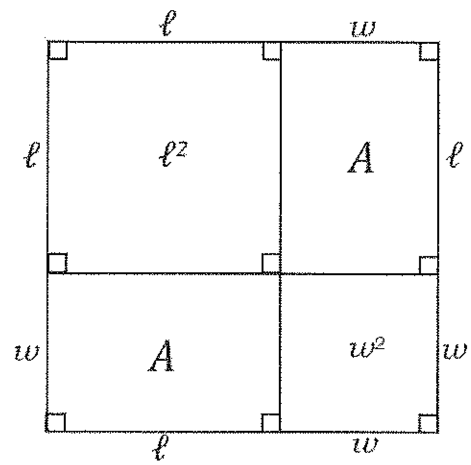


Figure 2

i. The area of the square with side length ℓ: ℓ^2

ii. The area of the square with side length w: w^2

iii. The area of the large square with side length $(\ell + w)$: $(\ell + w)^2$

iv. The area of the large square by adding areas: $\ell^2 + 2A + w^2$

Set (iii) and (iv) equal to each other:

$$\ell^2 + 2A + w^2 = (\ell + w)^2$$

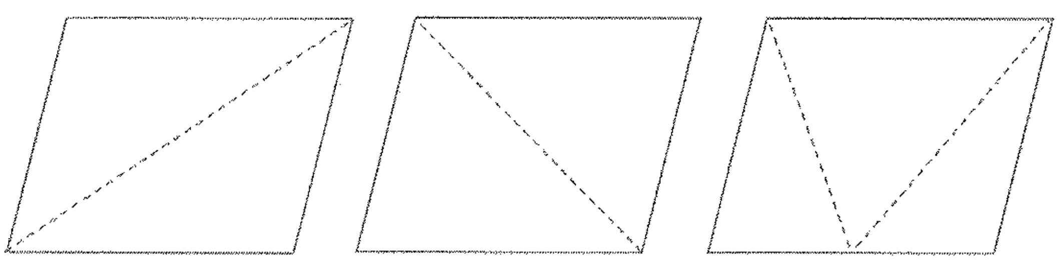
$$\ell^2 + 2A + w^2 = \ell^2 + 2\ell w + w^2$$

$$2A = 2\ell w$$

$$A = \ell w$$

3. Divide the following polygonal region into three different cases of the union of non-overlapping triangular regions.

Possible solution:



Lesson 3: The Scaling Principle for Area

1. Polygon X has a side length three times that of a corresponding side length of similar polygon Y. If the area of Y is 36, what is the area of X?

Length scale factor: 3

Area scale factor: $(3)^2$

$\text{Area}(X) = (3)^2 \times \text{Area}(Y)$

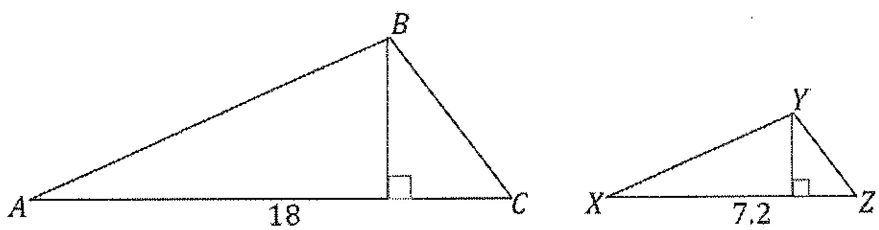
$\text{Area}(X) = (3)^2 \times 36$

$\text{Area}(X) = 324$

The area of polygon X *is* 324.

I must remember that if similar figures A and B are related by a scale factor of r, then their respective areas are related by a factor of r^2.

2. Triangles ABC and XYZ are similar. If the area of $\triangle ABC$ is 54, what is the area of $\triangle XYZ$?



Length scale factor: $\frac{18}{7.2} = 2.5$

Area scale factor: $(2.5)^2$

$\text{Area}(\triangle ABC) = (2.5)^2 \times \text{Area}(\triangle XYZ)$

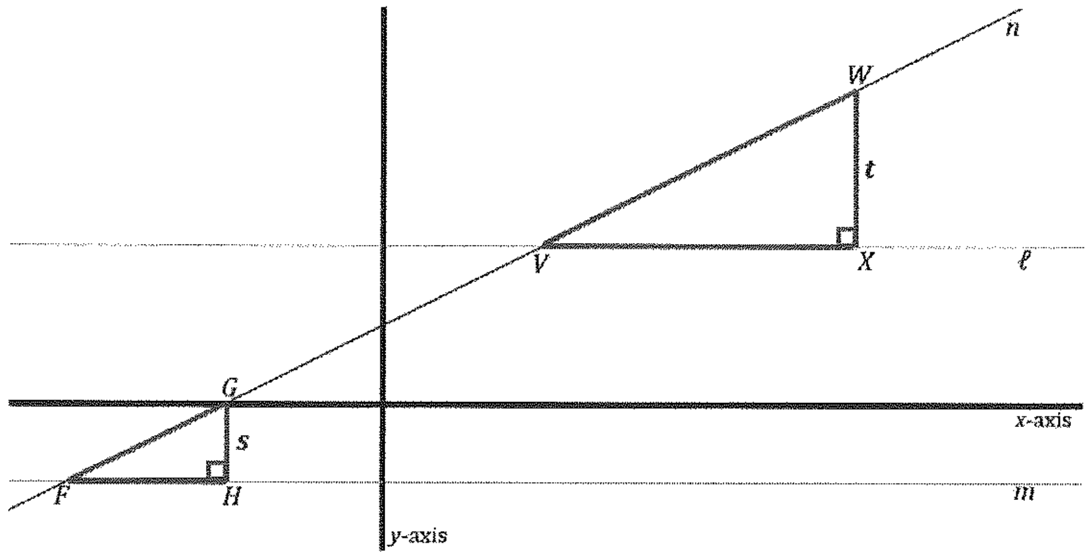
$54 = (2.5)^2 \times \text{Area}(\triangle XYZ)$

$54 = 6.25 \times \text{Area}(\triangle XYZ)$

$8.64 = \text{Area}(\triangle XYZ)$

The area of $\triangle XYZ$ *is* 8.64.

3. Triangles FGH and VWX are right triangles. Lines ℓ and m are parallel to the x-axis.



a. Are the triangles similar? Explain.

Yes. Both triangles are right triangles by construction, and since lines ℓ and m are parallel to the x-axis, they are also parallel to each other. The transversal line n creates corresponding angles, $\angle GFH$ and $\angle WVX$, of equal measure. Therefore, the triangles are similar by the AA criterion.

b. What is the scale factor that relates $\triangle FGH$ to $\triangle VWX$? What is the scale factor that relates $\triangle VWX$ to $\triangle FGH$?

The scale factor that relates $\triangle FGH$ to $\triangle VWX$ is $\frac{t}{s}$.

The scale factor that relates $\triangle VWX$ to $\triangle FGH$ is $\frac{s}{t}$.

c. State a relationship between the areas of the triangles and the scale factor that relates them.

$$\text{Area}(\triangle FGH) = \left(\frac{s}{t}\right)^2 \cdot \text{Area}(\triangle VWX)$$

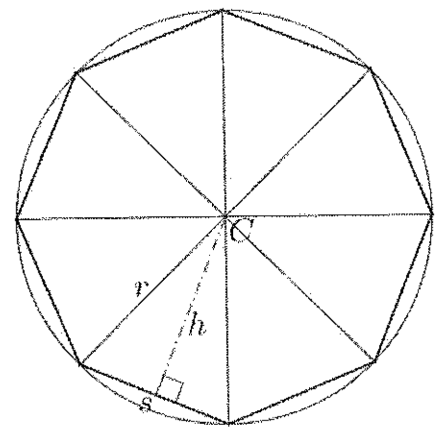
or

$$\text{Area}(\triangle VWX) = \left(\frac{t}{s}\right)^2 \cdot \text{Area}(\triangle FGH)$$

4. Describe the effect on the area of a triangle if
   a. The height were doubled and the base were doubled.
      The area would increase by a factor of 4.
   b. The height were doubled and the base were halved.
      The area would be unchanged.
   c. The height were cut to $\frac{1}{3}$ of the original height and the base were doubled.
      The area would decrease by a factor of $\frac{2}{3}$.
   d. The height were cut to $\frac{1}{4}$ of the original height and the base were cut to $\frac{1}{4}$ of the original base.
      The area would decrease by a factor of $\frac{1}{16}$.

Lesson 4: Proving the Area of a Disk

1. The following image is of a regular octagon inscribed in circle C with radius r. The distance from the center to each side s of the octagon is h.



   a. Write the formula for the area of the octagon in terms of the length of a side s and the distance from the center to each side, h.

   Area of one triangle formed by two radii and one side of the octagon: $\frac{1}{2}sh$

   The octagon can be divided into 8 such congruent triangles.

   Area of octagon: $8\left(\frac{1}{2}\right)sh$ ***or*** $4sh$

   b. The perimeter P of the octagon can be expressed as $P = 8s$. Rewrite the formula for the area of the octagon, and incorporate the expression of the perimeter into the formula.

   $$8\left(\frac{1}{2}\right)sh = 8s\cdot\left(\frac{h}{2}\right)$$
   $$= P\left(\frac{h}{2}\right)$$

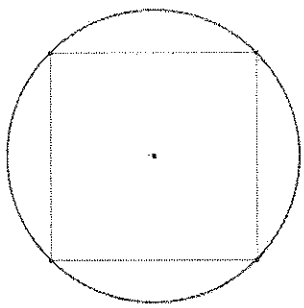
   I must remember that by the associative property of multiplication, a different arrangement of the factors reveals how P is expressed in the area formula of the octagon.

   c. Generalize this area formula in terms of perimeter for any regular polygon P_n with n sides of equal length inscribed in a circle.

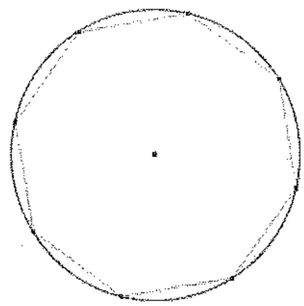
   $$n\left(\frac{1}{2}\right)sh = ns\cdot\left(\frac{h}{2}\right)$$
   $$= P_n\left(\frac{h}{2}\right)$$

2. Just as in prior lessons, we will use upper and lower approximations to help determine the area of a figure.

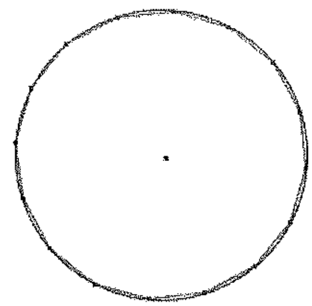
   a. The following images are of regular polygons, each inscribed in a circle with radius r and used to approximate the area of the disk that the circle defines. Inner polygons P_4, P_8, and P_{16} have 4, 8, and 16 sides, respectively.



P_4



P_8

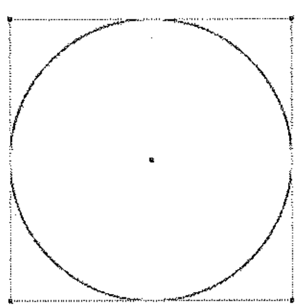


P_{16}

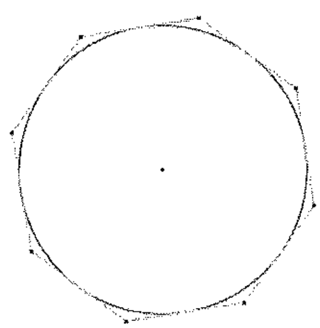
Do any of the polygons have an area that is greater than that of the disk? Which of the three polygons best approximates the area of the disk? Explain.

All of the polygons have areas that are less than that of the disk. Polygon P_{16} best approximates the area of the disk, as its area is greater than that of P_8 and P_4.

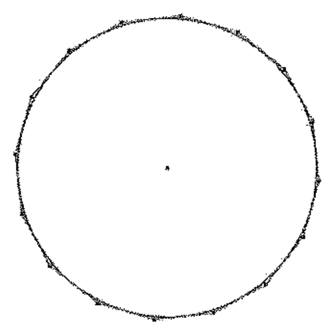
   b. The following images are of regular polygons, and each circumscribes a circle with radius r. Outer polygons P'_4, P'_8, and P'_{16} have 4, 8, and 16 sides, respectively.



P'_4



P'_8



P'_{16}

Do any of the polygons have an area that is less than that of the disk? Which of the three polygons best approximates the area of the disk? Explain.

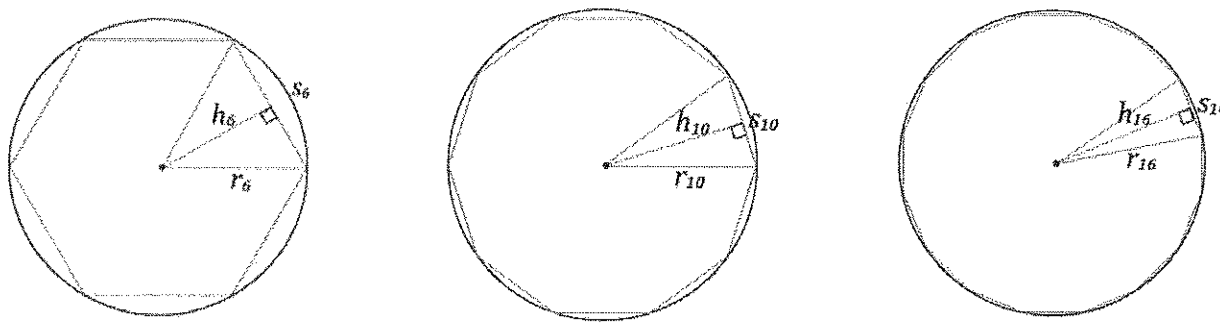
All of the polygons have areas that are greater than that of the disk. Polygon P'_{16} best approximates the area of the disk, as its area is less than that of P'_8 and P'_4.

A closer view of P'_{16} looks like this:

c. As the number of sides of the inner and outer polygons, n, increases, the lower and upper approximations of the area of the disk, $\text{Area}(P_n)$ and $\text{Area}(P'_n)$, respectively, become increasingly refined and more precise. Fill in the following inequality that describes the relative areas:

$$\textbf{Area}(P_n) < \text{Area(circle)} < \textbf{Area}(P'_n)$$

3. We define the area of the circle to be the number, or the limit, that the areas of the inner polygons converge to as n increases to infinity (or $n \to \infty$). Examine the following inner polygons.



a. What happens to h_n and $\text{Perimeter}(P_n)$ of polygon P_n as $n \to \infty$?

i. As n increases and approaches infinity, the height h_n approaches *the length of the radius* of the circle.

ii. As n increases and approaches infinity, $\text{Perimeter}(P_n)$ approaches *the circumference* of the circle.

I can see that the length of the height of each successive triangle is becoming more and more like the length of the radius. Furthermore, the perimeter of the inscribed polygon looks more and more like the circumference of the circle.

b. The area formula of an inscribed regular polygon in a circle is

$$\text{Area}(P_n) = [\text{Perimeter}(P_n)]\left(\frac{h_n}{2}\right).$$

Since we are defining the area of a circle as the limit of the areas of the inscribed regular polygon, substitute your answers for part (a) into the formulation for the area of a circle:

As $n \to \infty$, $h_n \to r$

As $n \to \infty$, $\textbf{Perimeter}(P_n) \to C$

$$\textbf{Area(circle)} = \frac{1}{2}r(2\pi r)$$

$$\textbf{Area(circle)} = \pi r^2$$

4. A circle and a regular hexagon each have a circumference and perimeter, respectively, of 20 units. Which figure has the greater area?

$C_{\text{circle}} = 24$; therefore, the radius is $\frac{12}{\pi}$.

$$\text{Area(circle)} = \pi\left(\frac{12}{\pi}\right)^2$$

$$\text{Area(circle)} = \frac{144}{\pi}$$

$$\text{Area(circle)} \approx 45.8$$

The area of the circle is approximately 45.8 units^2.

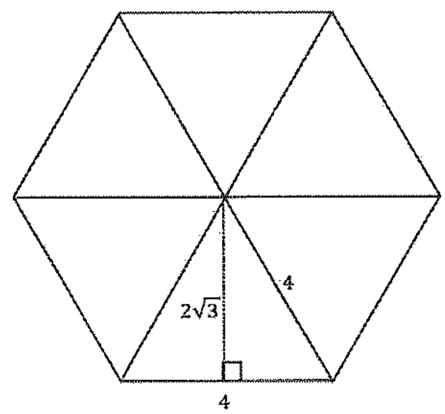
$P_{\text{hexagon}} = 24$; therefore, a side has length 4.

$$\text{Area(hexagon)} = 6 \cdot \frac{(4)(2\sqrt{3})}{2}$$

$$\text{Area(hexagon)} = 24\sqrt{3}$$

$$\text{Area(hexagon)} \approx 41.6$$

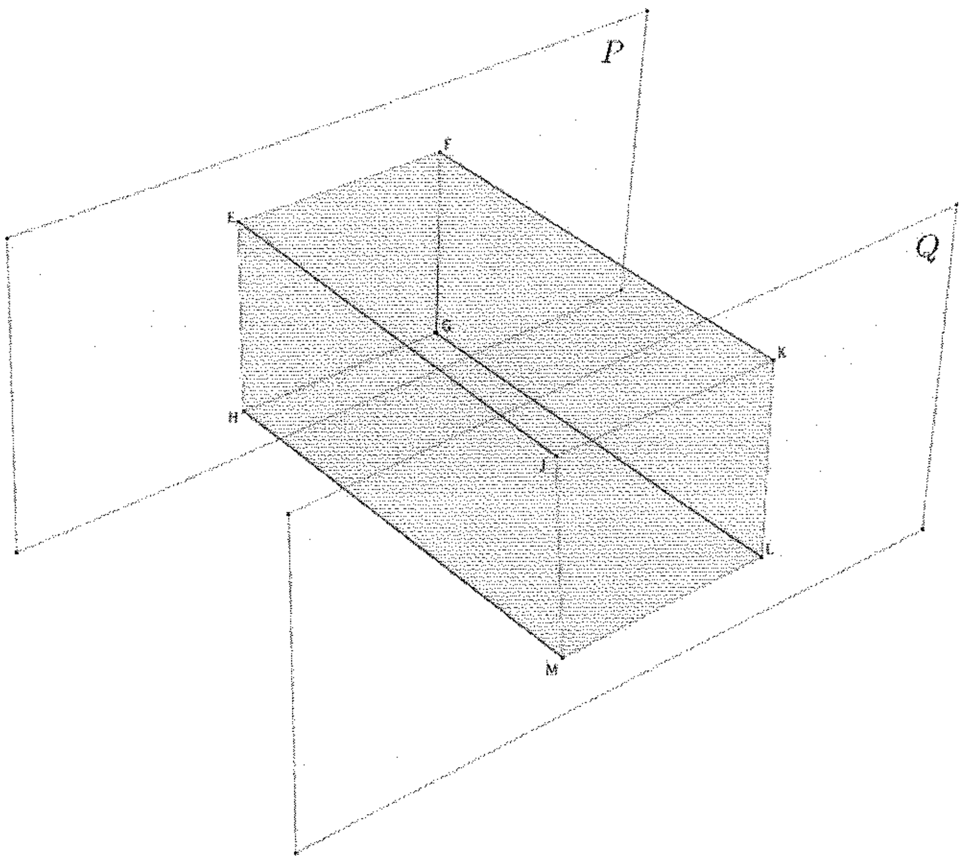
The area of the square is approximately 41.6 units^2.



Therefore, the circle has a greater area.

Lesson 5: Three-Dimensional Space

1. In the following figure, a right rectangular prism is determined by vertices $E, F, G, H, J, K, L,$ and M. Square bases $EFGH$ and $JKLM$ lie in planes P and Q, respectively. Which of the following statements are true?



a. $EJ = GL$ *True*

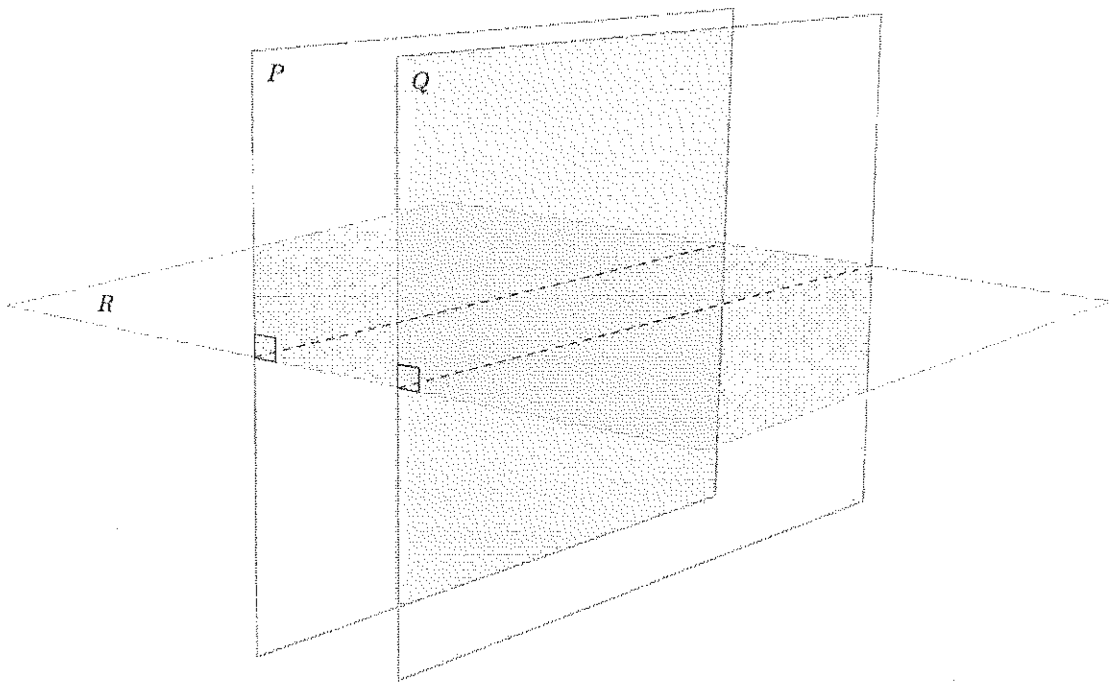
b. $\overline{HM}$ is perpendicular to planes P and Q. *True*

c. $EK = EL$ *False*

d. Planes P and Q are parallel. *True*

e. $ME = MG$ *True*

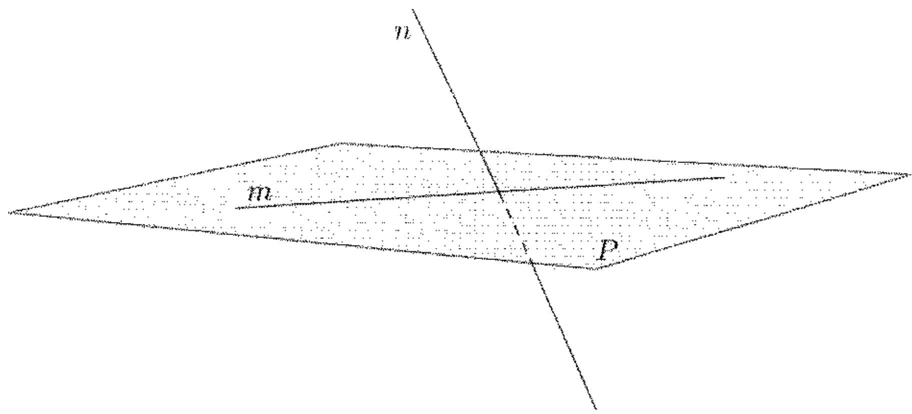
2. Sketch two parallel planes, P and Q. Sketch a third plane R that is perpendicular to P and Q. What do the intersections of the planes form?



The intersections form a pair of parallel lines.

I can sketch two parallel planes by showing that each of the planes is perpendicular to a third plane, ensuring that they are parallel to each other.

3. Sketch two lines, m and n. Sketch line m so that it lies in plane P, and sketch line n so that it intersects line m but passes through the plane.



Lesson 6: General Prisms and Cylinders and Their Cross-Sections

1. A general cylinder is defined as follows:

   General cylinder: (See Figure 1.) Let E and E' be two parallel planes, let B be a region in the plane E, and let L be a line that intersects E and E' but not B. At each point P of B, consider $\overline{PP'}$ parallel to L, joining P to a point P' of the plane E'. The union of all these segments is called a general cylinder with base B.

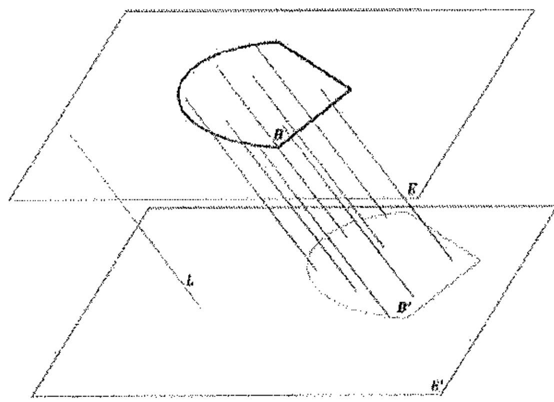
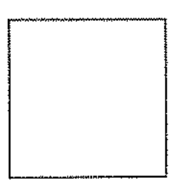
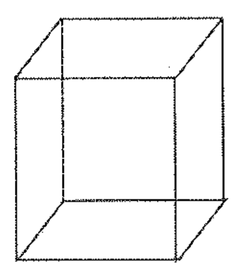


Figure 1

   A circular cylinder and a right rectangular prism are types of general cylinders. Describe the distinctions between the terms. How do prisms differ from general cylinders?

   In the definition of a right rectangular prism, the region B is a rectangular region; in the definition of a circular cylinder, the region B is a circular region. In the definition of a general cylinder, the shape of region B is not specified. Prisms are general cylinders with polygonal bases.

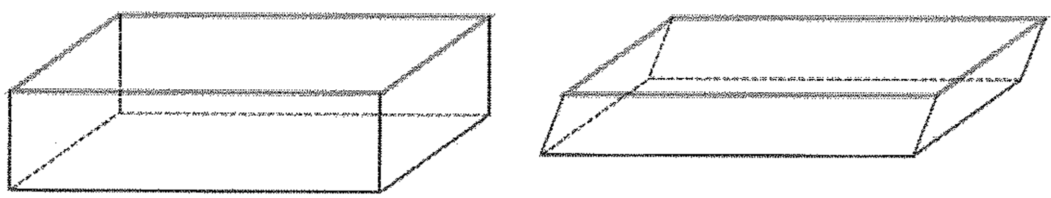
2. The following figures are cross-sections from their respective prisms. For each cross-section, sketch what its prism might look like:

   a.  

b.

c.

3. The following figures can each be used as one base of a rectangular prism. Sketch a right rectangular prism around the first figure and an oblique rectangular prism around the second.

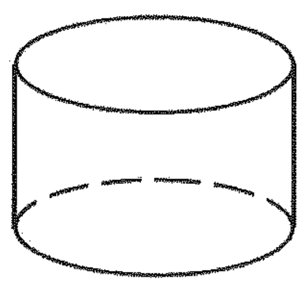


4. The following right cylinder has a lateral edge of 12 and a lateral area of 240π. What is the circumference and the radius of the base?

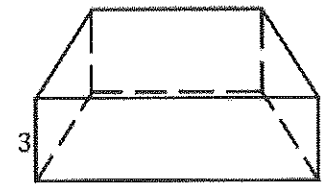
$$\text{lateral area} = (\text{circumference})(\text{lateral edge})$$
$$240\pi = (\text{circumference})(12)$$
$$20\pi = \text{circumference}$$

The circumference of the base is 20π.

The radius of the base is 10.



5. The following trapezoidal prism has a height of 3 units. If it has a total volume of 42 units^3, what is the area of one trapezoidal base?



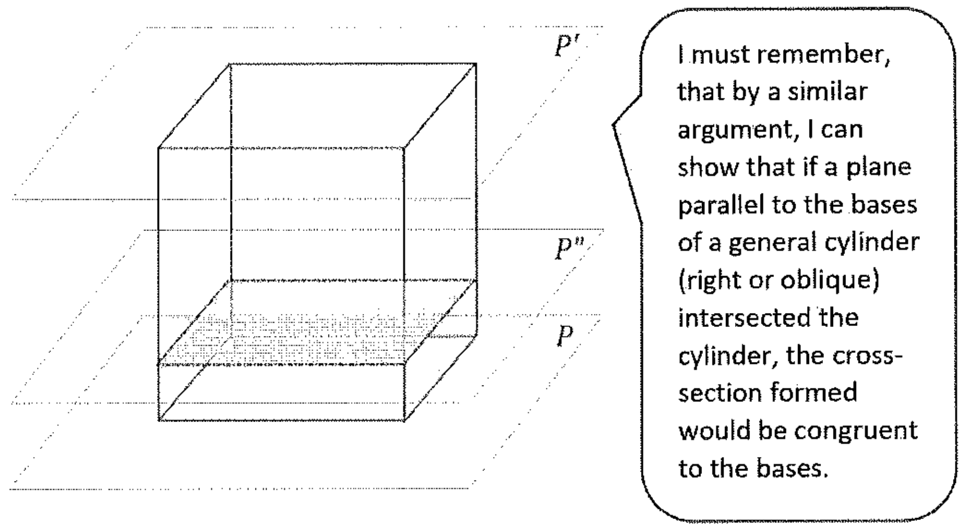
$\text{Volume} = (\text{Area of base}) \times (\text{height})$

$42 = (\text{Area of base}) \times (3)$

$14 = \text{Area of base}$

The area of one trapezoidal base is 14 units^2.

6. In the figure below, the bases of the rectangular prism lie in planes P and P'. A plane P'' is parallel to planes P and P'. What can be said about the cross-section formed by the intersection of the rectangular prism and plane P'' relative to the bases of the prism? Explain.



The cross-section formed by the intersection of the rectangular prism and plane P'' is congruent to the bases. The top portion of the prism, between planes P'' and P', is another prism and, hence, has congruent bases. Thus, the cross-section lying in P'' is congruent to both of the bases.

Lesson 7: General Pyramids and Cones and Their Cross-Sections

1. Pyramids and circular cones are types of general cones. Describe how each differs from the others.

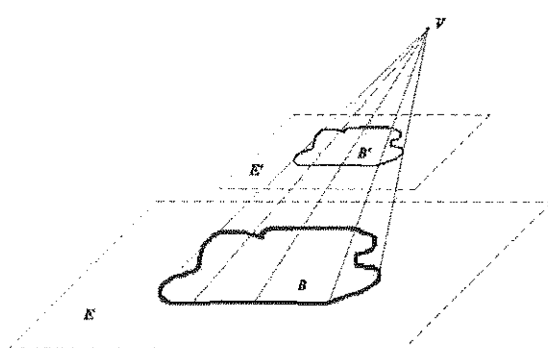
   We define a general cone as follows: Let B be a region in a plane E and V be a point not in E. The cone with base B and vertex V is the union of all $\overline{VP}$ for all points P in B. For a general cone, the shape of the region is not specified; however, a pyramid is a general cone with a polygonal base region, and a circular cone is a general cone with the base of a disk.

2. What distinguishes a right cone (or right pyramid) from a general cone?

   A general cone whose vertex lies on the perpendicular line to the base and that passes through the center of the base is said to be a right cone.

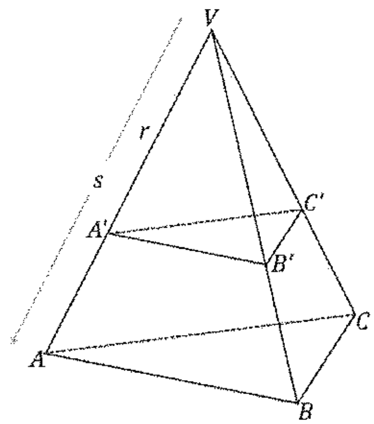
I should remember the parallel that exists between general cylinders, prisms, and circular cylinders and general cones, pyramids, and circular cones.

3. What argument supports that a cross-section of a general cone is similar to the base of the cone?



Dilations behave in three-dimensional space similarly to the way they behave in two-dimensional space. We can show the existence of a dilation that maps the base B to region B' (see the figure above); thus this similarity transformation implies that the cross-section is similar to the base.

4. In the following triangular pyramid, a plane passes through the pyramid so that it is parallel to the base and results in the cross-section $\triangle A'B'C'$.



a. Use similarity criteria to explain why $\triangle A'B'C'$ must be similar to $\triangle ABC$.

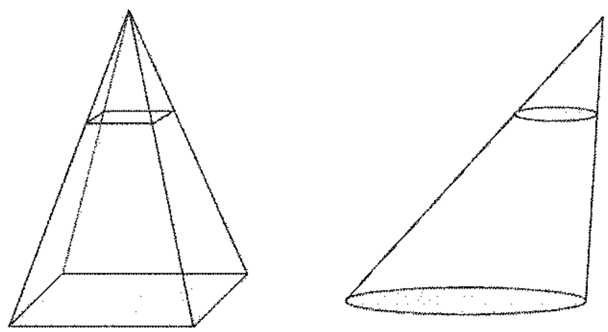
Since the plane that intersects the pyramid is parallel to the base, the edges of triangle $\triangle A'B'C'$ are parallel to the corresponding edges of $\triangle ABC$. By the triangle side splitter theorem, each of the segments of $\triangle A'B'C'$ proportionally split triangles VAB, VBC, and VCA. A dilation with center V maps A to A', B to B', and C to C' by scale factor $\frac{r}{s}$. Since each of the lengths of $\triangle A'B'C'$ are $\frac{r}{s}$ the lengths of $\triangle ABC$, the triangles are similar by the SSS similarity criterion.

> Recall that the triangle side splitter theorem states that a line segment splits two sides of a triangle proportionally if and only if it is parallel to the third side.

b. Write a statement that shows the relationship in area between $\triangle ABC$ and $\triangle A'B'C'$.

$$\text{Area}(\triangle A'B'C') = \left(\frac{r}{s}\right)^2 \text{Area}(\triangle ABC)$$

5. The following right square pyramid and general cone have heights of equal distance and bases that are equal in area. A cross-section is taken $\frac{1}{4}$ of the perpendicular distance between each vertex and the respective base of each cone. If the area of the circular cone's cross-section is 6.25, what is the side length of the base of the pyramid?



The distance from the cross-section to the vertex is $\frac{1}{4}$ the height of each cone, so the scale factor from the cross-section to the base is 4 (for either cone). The areas of the planar regions are related by the square of the scale factor, or 16.

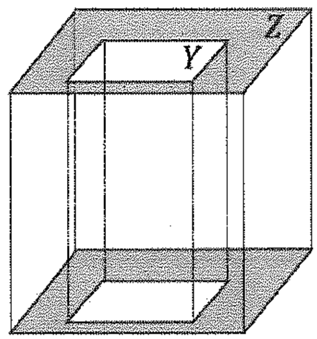
$$\text{Area(base)} = 16(\text{Area(cross-section)})$$
$$\text{Area(base)} = 16(6.25)$$
$$\text{Area(base)} = 100$$

Since the side length of the square base is the square root of its area, the side length is $\sqrt{100}$, or 10.

By the general cone cross-section theorem, if two general cones have the same base area and the same height, then cross-sections for the cones have the same distance from the vertex and the same area.

Lesson 8: Definition and Properties of Volume

1. A machine part is made by making a "hole" in the shape of a rectangular prism in a larger rectangular prism, like this:



Use notation to describe the volume of the part after the hole, Y, has been taken out.

$\mathbf{Vol}(Z - Y) = \mathbf{Vol}(Z) - \mathbf{Vol}(Y)$

2. A right triangular prism, P, with height h and triangular base T, has two sub-triangular prisms, P_1 and P_2, with triangular bases T_1 and T_2.

   a. Describe why the volume formula of P is $\text{Area}(T) \cdot h$.

   Each of the sub-triangular prisms has a volume formula of Ah. The height is the same for both sub-triangular prisms; therefore, the volume of prism P can be found by taking the sum of the areas of the bases of the sub-triangular prisms, or $\text{Area}(T)$, and multiplying it by the height of the prism.

   b. The volume of the union of two solids is the sum of the volumes minus the volume of the intersection. Rewrite the following equation to show how the volume of the right triangular prism P is equal to the area of base T multiplied by the height, h.

   $\text{Vol}(\text{P}) = \text{Vol}(\text{P}_1) + \text{Vol}(\text{P}_2) - \text{Vol}(\text{rectangle})$

   $\mathbf{Vol}(P) = \left(\mathbf{Area}(T_1) \cdot h\right) + \left(\mathbf{Area}(T_2) \cdot h\right) - 0$

   $\mathbf{Vol}(P) = \mathbf{Area}(T_1 + T_2) \cdot h$

   $\mathbf{Vol}(P) = \mathbf{Area}(T) \cdot h$

   This volume property has an analogous area property: The area of the union of two regions is the sum of the areas minus the area of the intersection.

3. In Lesson 1, we estimated the area of a region by tiling the region with complete rectangles and triangles in a way that created upper and lower approximations of the area. What connection can be made between this estimation process with areas and the estimation of the volume of a general right cylinder? What is the volume formula of a general right cylinder?

   To determine the volume of a general right cylinder with base B, we approximate the area of B in the same way that we did in Lesson 1. Since we know the area formula for rectangular and triangular prisms, we multiply each of the rectangular and triangular base areas by the same height, or we take the total area of the base and multiply by the height. This process creates lower and upper approximations. The actual value of the volume of the general right cylinder will fall in between these approximations. We conclude that the volume of a general right cylinder is the area of base, B, multipled by the height, h.

4. The dimensions of a commonly produced 1-gram gold bar is approximately $16 \text{ mm} \times 8 \text{ mm} \times 0.4 \text{ mm}$. What is the density, in grams per cubic centimeters, of gold based on these dimensions?

   The density of a substance can be found using the formula:
   $$\text{density} = \frac{\text{mass}}{\text{volume}}.$$

   $\mathbf{Volume(bar) = 16\ mm \times 8\ mm \times 0.4\ mm}$
   $\mathbf{Volume(bar) = 51.2\ mm^3}$

   $\mathbf{Volume(bar)}$ ***in cubic centimeters:***

   $$\mathbf{51.2\ mm^3 \left(\frac{0.001\ cm^3}{1\ mm^3}\right) = 0.512\ cm^3}$$

   $\mathbf{Density(gold)}$***:***

   $$\mathbf{\left(\frac{1\ g}{0.512\ cm^3}\right) \approx 19.5\ g/cm^3}$$

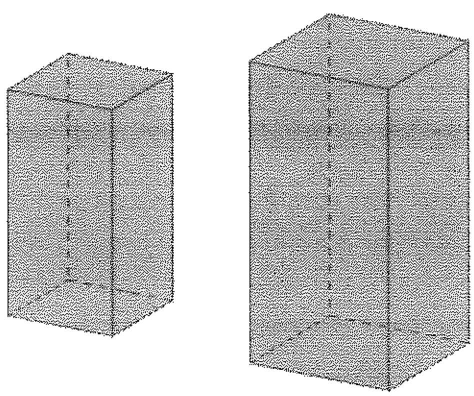
EUREKA MATH

Lesson 9: Scaling Principle for Volumes

1. Two similar right rectangular prisms have bases with areas in the ratio $9:16$. If the volume of the smaller rectangular prism is 36 cm^3, find the volume of the larger rectangular prism.

If the ratio of the base areas of the prisms is $9:16$, then the area of the base of the larger prism is $\frac{16}{9}$ times the area of the smaller prism. Areas of similar figures are related by the square of the scale factor that relates the figures, so the scale factor is $\sqrt{\frac{16}{9}} = \frac{4}{3}$.

I don't know the dimensions of the bases, so I cannot use them to find the scale factor of the larger prism. If the prisms are similar, then their bases are similar, and I remember that areas of similar figures are related by the square of the scale factor. I can use the ratio of the base areas to find the scale factor.



Since the areas of similar figures are related by the square of the scale factor, I can use the square root of the value of $\frac{16}{9}$ to find the scale factor.

Volumes of similar three-dimensional figures are related by the cube of the scale factor, so the volume of the larger prism is $\left(\frac{4}{3}\right)^3$ times the volume of the smaller prism, or $\frac{64}{27}$ times the volume of the smaller prism.

$$\frac{64}{27}(36\text{ cm}^3) = 85\frac{1}{3}\text{ cm}^3$$

The volume of the larger prism is $85\frac{1}{3}\text{ cm}^3$.

2. The volume of a general cylinder is 176 m^3. A similar general cylinder has a volume of $42\frac{31}{32} \text{ m}^3$. Calculate the value of the ratio of the similar cylinders' corresponding lengths.

For two similar figures whose corresponding lengths are in the ratio $a:b$, the ratio of their volumes is $a^3:b^3$. Then the volume of the smaller general cylinder is $\frac{42\frac{31}{32}}{176}$ times the volume of the larger general cylinder.

Let r be the scale factor that relates the two cylinders.

$$r^3 = \frac{42\frac{31}{32}}{176}$$

$$\sqrt[3]{r^3} = \sqrt[3]{\frac{42\frac{31}{32}}{176}}$$

$$r = 0.625 = \frac{5}{8}$$

> This value relates the volumes of the similar general cylinders, so it must be the cube of the scale factor. I think I will use my calculator to find the cube root of this complex fraction.

> The value of the ratio of corresponding side lengths of similar figures is equal to the scale factor relating the figures. I have to calculate the scale factor first and then write a ratio with the same value.

The scale factor that relates the smaller cylinder to the larger cylinder is $\frac{5}{8}$. Since the ratio of any corresponding dimensions must have the same value as the scale factor, the value of the ratio of corresponding lengths must be $5:8$.

3. The height of a pyramid is $\frac{3}{4}$ the height of a similar pyramid. If the volume of the smaller pyramid is $1{,}000$ cubic units, what is the volume of the larger pyramid?

Let h represent the height of the larger pyramid. Then the height of the smaller pyramid is $\frac{3}{4}h$. The scale factor of the similarity transformation that takes the larger pyramid to the smaller pyramid is $\frac{3}{4}$. The volumes of similar figures are related by the cube of the scale factor.

> If I use h to represent the height of the larger pyramid, then the height of the smaller pyramid is $\frac{3}{4}h$. By the value of the ratio of corresponding heights, the scale factor from the larger pyramid to the smaller pyramid is $\frac{3}{4}$.

$$\text{Volume(smaller pyramid)} = \left(\frac{3}{4}\right)^3 \cdot \text{Volume(larger pyramid)}$$

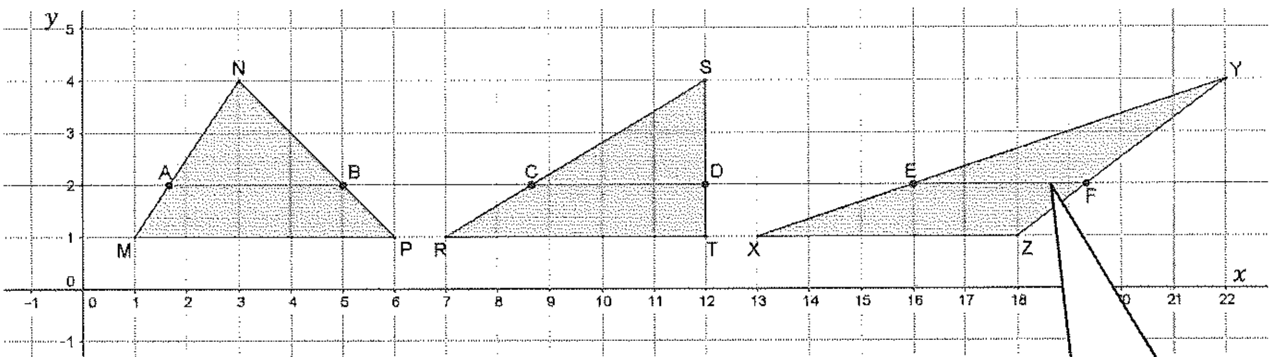
$$1000 = \frac{27}{64} \cdot \text{Volume(larger pyramid)}$$

$$2370.37 \approx \text{Volume(larger pyramid)}$$

The volume of the larger pyramid is approximately 2370.37 cubic units.

Lesson 10: The Volumes of Prisms and Cylinders and Cavalieri's Principle

1. In the diagram below, the line $y = 2$ intersects the triangles as shown. Show that $AB = CD = EF$.



$\overline{MN}$ lies on the line with equation $y - 1 = \frac{3}{2}(x - 1)$. The y-coordinate of A is 2.

$$2 - 1 = \frac{3}{2}(x - 1)$$
$$1 = \frac{3}{2}(x - 1)$$
$$\frac{2}{3} = x - 1$$
$$\frac{5}{3} = x$$

To find the length of the line segment intersecting a triangle on the coordinate plane, I have to find the coordinates of the points where the line intersects the sides of the triangle. I have to find the equations of the lines that contain the sides of the triangles.

The coordinates of point A are $\left(\frac{5}{3}, 2\right)$.

$\overline{NP}$ lies on the line with equation $y - 1 = -1(x - 6)$. The y-coordinate of B is 2.

$$2 - 1 = -1(x - 6)$$
$$1 = -1(x - 6)$$
$$-1 = x - 6$$
$$5 = x$$

The coordinates of point B are $(5, 2)$.

$$AB = \sqrt{\left(5 - \frac{5}{3}\right)^2 + (2-2)^2}$$

Now I can use the distance formula with the coordinates of the endpoints that I found.

$$AB = \sqrt{\left(\frac{15}{3} - \frac{5}{3}\right)^2 + 0^2}$$

$$AB = \sqrt{\left(\frac{10}{3}\right)^2}$$

$$AB = \frac{10}{3}$$

The length of the portion of the line $y = 2$ that intersects triangle MNP is $\frac{10}{3}$.

$\overline{RS}$ lies on the line with equation $y - 1 = \frac{3}{5}(x - 7)$. The y-coordinate of R is 2.

Now I will use the same process to find CD and EF.

$$2 - 1 = \frac{3}{5}(x - 7)$$

$$1 = \frac{3}{5}(x - 7)$$

$$\frac{5}{3} = x - 7$$

$$\frac{5}{3} + 7 = x$$

$$\frac{26}{3} = x$$

The coordinates of point C are $\left(\frac{26}{3}, 2\right)$.

$\overline{ST}$ lies on the line with equation $x = 12$, so the coordinates of point D are $(12, 2)$.

$$CD = \sqrt{\left(12 - \frac{26}{3}\right)^2 + (2-2)^2}$$

$$CD = \sqrt{\left(\frac{36}{3} - \frac{26}{3}\right)^2 + 0^2}$$

$$CD = \sqrt{\left(\frac{10}{3}\right)^2}$$

$$CD = \frac{10}{3}$$

The length of the portion of the line $y = 2$ that intersects triangle RST is $\frac{10}{3}$, so it follows that $AB = CD$.

The coordinates of point E are $(16, 2)$.

$\overline{ZY}$ lies on the line with equation $y - 1 = \frac{3}{4}(x - 18)$. The y-coordinate of F is 2.

$$2 - 1 = \frac{3}{4}(x - 18)$$
$$1 = \frac{3}{4}(x - 18)$$
$$\frac{4}{3} = x - 18$$
$$\frac{4}{3} + 18 = x$$
$$\frac{58}{3} = x$$

The coordinates of point F are $\left(\frac{58}{3}, 2\right)$.

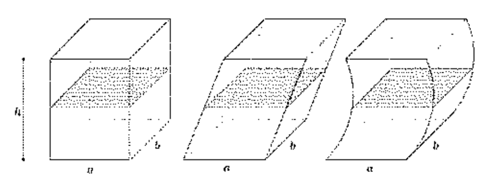
$$EF = \sqrt{\left(\frac{58}{3} - 16\right)^2 + (2 - 2)^2}$$
$$EF = \sqrt{\left(\frac{58}{3} - \frac{48}{3}\right)^2 + 0^2}$$
$$EF = \sqrt{\left(\frac{10}{3}\right)^2}$$
$$EF = \frac{10}{3}$$

The portion of the length of the line $y = 2$ that intersects triangle XYZ is $\frac{10}{3}$, so it follows that $AB = CD = EF$.

2. Why do a right circular cylinder and an oblique circular cylinder, both having bases with radii of r and heights of h, have the same volume?

   The cross-sections of a general cylinder are identical. If the cross-sections of the oblique circular cylinder are repositioned so that their centers lie on a line perpendicular to the base, then the resulting solid would be a right circular cylinder. Since the resulting solid is made up of all the same cross-sections as the original oblique cylinder, their volumes must be the same.

I saw in the lesson that a stack of paper is the same stack of paper whether it looks like a right prism or an oblique prism. If the bases and heights are the same, the volumes are the same.



3. Use Cavalieri's principle to explain why a circular cylinder with a base of radius 9 and a height of 4 has the same volume as a rectangular prism whose base has side lengths of 9 and 9π and has a height of 4.

The area of the base of the circular cylinder:

$$\text{Area} = \pi \cdot r^2$$
$$\text{Area} = \pi \cdot 9^2$$
$$\text{Area} = 81\pi$$

The area of the base of the rectangular prism:

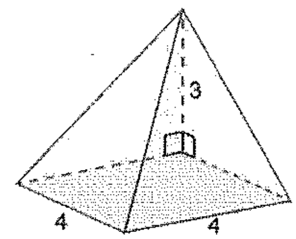
$$\text{Area} = \text{length} \cdot \text{width}$$
$$\text{Area} = 9 \cdot 9\pi$$
$$\text{Area} = 81\pi$$

Cavalieri's principle is based on the areas of cross-sections of solids. The given solids are both general cylinders, and I know that all cross-sections in a general cylinder are identical, so I just need to show that the areas of the bases of the solids are equal.

The areas of the bases of the circular cylinder and the rectangular prism are both 81π. All cross-sections of a general cylinder made parallel to the base of the cylinder have the same area as the base. This means that every cross-section of the circular cylinder has an area of 81π. Since a prism is a general cylinder, it is also true that every cross-section of the prism made parallel to the base of the prism has an area of 81π. Each of the given solids has a height of 4; thus, both solids have cross-sections of equal areas at every height. By Cavalieri's principle, the two solids must have the same volume.

Lesson 11: The Volume Formula of a Pyramid and Cone

1. Find the volume of the rectangular pyramid shown to the right.



$\text{Volume} = \frac{1}{3}B \cdot h$

$\text{Volume} = \frac{1}{3}(l \cdot w)$

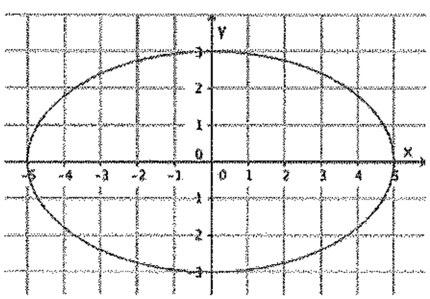
$\text{Volume} = \frac{1}{3}(4 \cdot 4) \cdot 3$

$\text{Volume} = 16$

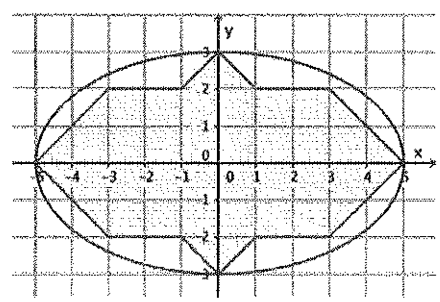
The volume of a pyramid is $\frac{1}{3}$ of the volume of a prism that has the same base and height.

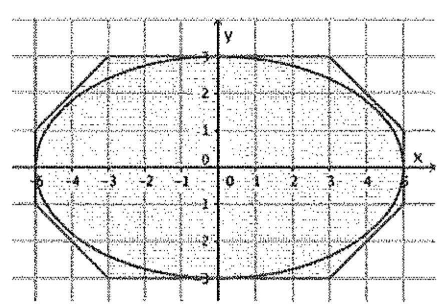
The volume of the rectangular pyramid is 16.

2. In an earlier lesson, the area of an ellipse was approximated by averaging approximations from inner and outer polygons. A given general cone has the elliptical base shown in the image and has a height of 9 units. Use the average of the approximations of the base area to find the approximate volume of the cone.



The volume formula for a general cone is $V = \frac{1}{3}Bh$, where B is the area of the base of the cylinder and h is the height. I need to start by approximating the upper and lower areas of the base and then average them.





Answers may vary depending on the size of squares and right triangles used to compose the approximating polygons.

A lower approximation of the area of the elliptical base is 34 *square units. An upper approximation of the area of the base is* 52 *square units. The average of the approximations is* 43 *square units.*

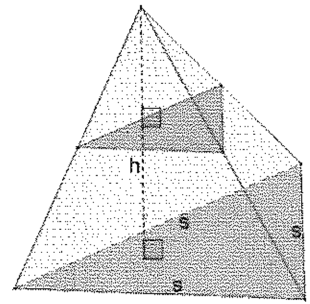
$\text{Volume} = \frac{1}{3}B \cdot h$

$\text{Volume} \approx \frac{1}{3}(43)(9)$

$\text{Volume} \approx 129$

The approximate volume of the general cone is 129 *cubic units.*

3. A triangular pyramid has an equilateral base with side lengths of s and a height of h. The pyramid is sliced along a cross-section parallel to its base at a height of $\frac{1}{2}h$. Calculate the volumes to show how much larger the volume of the solid below the cross-section is than the volume of the solid above the cross-section.



The section of the solid that is above the cross-sectional cut is similar to the original pyramid by a dilation centered at the apex of the pyramid with a scale factor of $\frac{1}{2}$.

The volume of the original pyramid, V, requires the area of the equilateral triangle base, B. The height of the triangle must be calculated using the Pythagorean theorem.

$h = \frac{1}{2}s \cdot \sqrt{3}$ *The ratio of side lengths in a 30-60-90 triangle is $1:\sqrt{3}:2$.*

If I draw one altitude in an equilateral triangle, the altitude divides the triangle into two 30–60–90 right triangles. I can use the ratio of the side lengths to calculate the height of the triangle and the area of the base of the pyramid.

$$B = \frac{1}{2}bh$$

$$B = \frac{1}{2} \cdot s \cdot \left(\frac{1}{2} \cdot s\sqrt{3}\right)$$

$$B = \frac{s^2\sqrt{3}}{4}$$

$$V = \frac{1}{3}B \cdot h$$

$$V = \frac{1}{3}\left(\frac{s^2\sqrt{3}}{4}\right)h$$

$$V = \frac{s^2h\sqrt{3}}{12}$$

The volume of the original pyramid is $\frac{s^2h\sqrt{3}}{12}$.

By the scaling principle for volume, the volume of the similar pyramid, V_s, that lies above the cross-section is the cube of the scale factor times the volume of the original pyramid.

$$V_s = \left(\frac{1}{2}\right)^3 \cdot \frac{s^2h\sqrt{3}}{12}$$

$$V_s = \frac{1}{8} \cdot \frac{s^2h\sqrt{3}}{12}$$

$$V_s = \frac{s^2h\sqrt{3}}{96}$$

I have multiple volumes to work with, so I should use some subscripts to keep them organized.

The volume of the similar pyramid that lies above the cross-section is $\frac{s^2h\sqrt{3}}{96}$.

The volume of the original pyramid that lies below the cross-section, V_b, is the difference of the volumes.

$$V_b = V - V_s$$
$$V_b = \frac{s^2h\sqrt{3}}{12} - \frac{s^2h\sqrt{3}}{96}$$
$$V_b = \frac{8s^2h\sqrt{3}}{96} - \frac{s^2h\sqrt{3}}{96}$$
$$V_b = \frac{7s^2h\sqrt{3}}{96}$$

The volume of the pyramid that lies below the cross-section is $\frac{7s^2h\sqrt{3}}{96}$.

The value of the ratio of the volume of the pyramid below the cross-section to the volume of the pyramid above the cross-section is

If I want to know how the volumes compare, I can use a ratio and the value of the ratio to show the relationship.

$$\frac{\frac{7s^2h\sqrt{3}}{96}}{\frac{s^2h\sqrt{3}}{96}} = \frac{7s^2h\sqrt{3}}{96} \cdot \frac{96}{s^2h\sqrt{3}} = 7$$

The volume of the pyramid that lies below the cross-section is 7 times the volume of the pyramid that lies above the cross-section.

Note: Without calculating the volumes, this question could be answered using the scaling principle for volume directly.

Lesson 12: The Volume Formula of a Sphere

1. Find the volume of a solid sphere, V, that has a radius of $5\frac{1}{2}$ cm.

$V = \frac{4}{3}\pi r^3$

$V = \frac{4}{3}\pi\left(5\frac{1}{2}\right)^3$

$V = \frac{4}{3}\pi\left(\frac{11}{2}\right)^3$

$V = \frac{4}{3}\pi\left(\frac{1331}{8}\right)$

$V = \frac{1331}{6}\pi \approx 696.9$

The volume of the solid sphere is approximately 696.9 cm^3.

The volume formula for a sphere is $V = \frac{4}{3}\pi r^3$, where r represents the radius of the sphere.

It's often easier to work with mixed numbers if they are converted into a fraction greater than one.

2. The sphere in Problem 1 has a spherical cavity removed from its center. The remaining shell has a uniform wall thickness of 2 cm. Find the volume of the spherical cavity, V_c.

The wall thickness makes up 2 cm *of the radius of the original sphere, so the cavity must have a radius of* $3\frac{1}{2}\text{ cm}$.

To find the volume of the spherical cavity, I need to know its radius. The thickness of the shell is 2 cm, so the radius of the cavity must be 2 cm less than the radius of the original solid sphere.

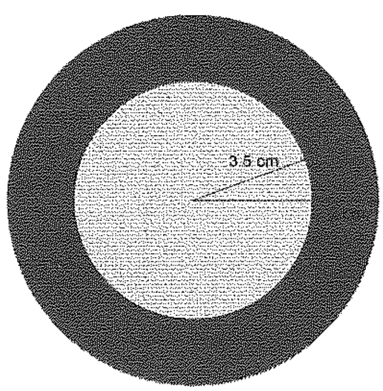
$V_c = \frac{4}{3}\pi r^3$

$V_c = \frac{4}{3}\pi\left(3\frac{1}{2}\right)^3$

$V_c = \frac{4}{3}\pi\left(\frac{7}{2}\right)^3$

$V_c = \frac{4}{3}\pi\left(\frac{343}{8}\right)$

$V_c = \frac{343}{6}\pi \approx 179.6$



The volume of the spherical cavity is approximately 179.6 cm^3.

3. What is the volume of material that lies outside the spherical cavity, V_m?

The volume of the material that lies outside the spherical cavity, V_m, is equal to the difference of the total volume of the solid sphere, V, and the volume of the spherical cavity, V_c.

$V_m = V - V_c$

$V_m = \frac{1331}{6}\pi - \frac{343}{6}\pi$

$V_m = \frac{988}{6}\pi \approx 517.3$

To obtain the most accurate answer, I should subtract the exact volumes that I calculated in Problems 1 and 2. Using the approximate values could give me a significant rounding error.

The volume of the material that lies outside the spherical cavity is approximately $517.3\ \text{cm}^3$.

Lesson 13: How Do 3D Printers Work?

1. A solid sphere is to be printed on a 3D printer with a radius of 15 units.
   a. If the object will be printed using eight equally thick layers of print media, what must be the thickness of each layer?

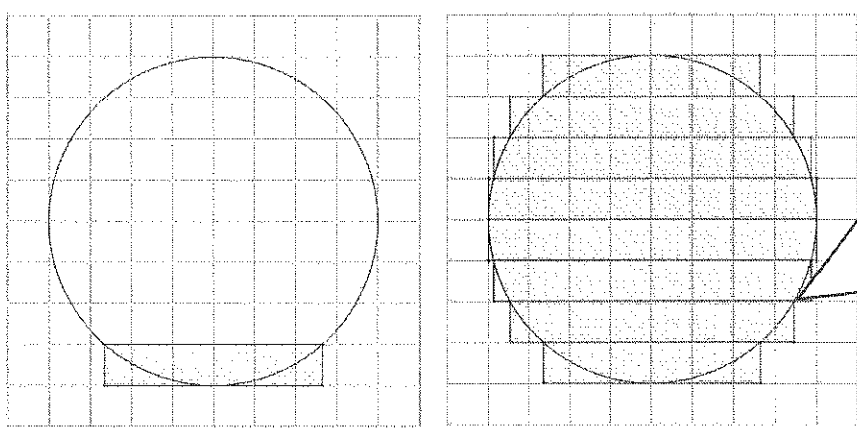
   With a radius of 15 units, the greatest width and total height of the printed solid must be 30 units.

   $$\frac{30}{8} = 3\frac{3}{4}$$

   Divided into eight equally thick layers, each layer must be $3\frac{3}{4}$ units thick.

   The radius is only half of the height and diameter of the spherical object, so I have to double the radius before I determine the thickness of the printed layers.

   b. The first layer of the printed solid is drawn in the grid below from a side view. Draw the remaining layers that will give the closest approximation of the sphere.



   I need to look for the point of intersection of the sphere with the top of each new layer. If the layers are not wide enough, the sphere will be incomplete.

   c. Noah, the printer, plans to produce several printed spheres using the 3D printer. Are there any modifications that can be made to the above design to conserve print media? Explain.

   The printer could conserve print media by removing some of the material from the centers of layers two through seven, leaving a hollow core inside the sphere. Material cannot be removed from layers one or eight because this would result in holes in the outer surface of the printed sphere.

   If I want to conserve material, I need to look for places where material can be removed without affecting the structure of the resulting figure.

d. After producing one printed sphere, Noah determines that his product is not a close enough approximation of a sphere because he cannot easily roll it. How could Noah design a closer approximation of a sphere?

Noah's printed sphere has only eight layers so the top and bottom of the printed figure are fairly large, flat surfaces. If Noah can produce the printed figure using a greater number of layers, each layer would be a closer approximation to a layer within the sphere and would also have less waste.

> Noah's design does not have enough layers to look much like a sphere. In order to look more round, the layers would have to be much thinner and closer in diameter. In order to make thinner layers, I would have to make many more layers.

2. A circular cone with a base radius of 13.7 millimeters and a height of 11 millimeters is to be printed on a 3D printer. The printer filament to be used is cylindrical and has a diameter of 1.75 millimeters. There is a length of approximately 1 meter left on the spool. Will there be enough filament to print the circular cone?

Let V_c represent the volume of the cone to be approximated.

$$V_c = \frac{1}{3}\pi r^2 \cdot h$$

$$V_c = \frac{1}{3}\pi(13.7\text{ mm})^2 \cdot 11\text{ mm}$$

$$V_c = \frac{1}{3}\pi \cdot 187.69\text{ mm}^2 \cdot 11\text{ mm}$$

$$V_c = \frac{2064.59\pi}{3}\text{ mm}^3 \approx 2162.0\text{ mm}^3$$

> 3D printers do not produce exact 3D objects because they print in layers. I can determine the approximate volume of filament needed by calculating the volume of the exact circular cone.

The volume of the actual cone is approximately $2,162.0\text{ mm}^3$; however, the printed figure will be an approximation of the cone and will, therefore, contain slightly more material.

Let V_f represent the volume of filament available for the print job.

$$1\text{ m} = 1000\text{ mm}$$

$$V_f = \pi r^2 \cdot h$$

$$V_f = \pi\left(\frac{1.75\text{ mm}}{2}\right)^2 \cdot 1000\text{ mm}$$

$$V_f = 765.625\pi\text{ mm}^3 \approx 2405.3\text{ mm}^3$$

> If the filament is cylindrical, I can find the volume of material left on the spool using the volume formula for a circular cylinder.

The volume of filament left on the spool is approximately $2,405.3\text{ mm}^3$, which is about 243.3 mm^3 more than the amount needed to print an exact circular cone. If the printed approximation of the cone is a close enough approximation, there should be enough filament left for the print job.