Homework Helpers

Geometry Module 2

Lesson 1: Scale Drawings

1. What is a well-scaled drawing?

   A well-scaled drawing of a figure is one where corresponding angles are equal in measure, and corresponding lengths are all in the same proportion.

2. Triangle DEF is a scale drawing of triangle ABC with scale factor r. Describe each of the following statements as sometimes, always, or never true, and justify your response.

   > I should remember that a scale drawing of a figure is made without respect to the orientation of the original figure.

   a. $DE > AB$

   Sometimes but only when $r > 1$

   b. $m\angle E < m\angle B$

   Never because corresponding angles between scale drawings are equal in measure

   c. $\frac{EF}{DF} = \frac{BC}{AC}$

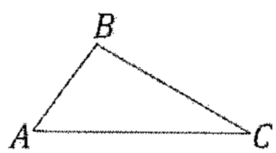
   Always because distances in a scale drawing are equal to their corresponding distances in the original drawing times the scale factor: $\frac{EF}{DF} = \frac{r(EF)}{r(DF)} = \frac{BC}{AC}$

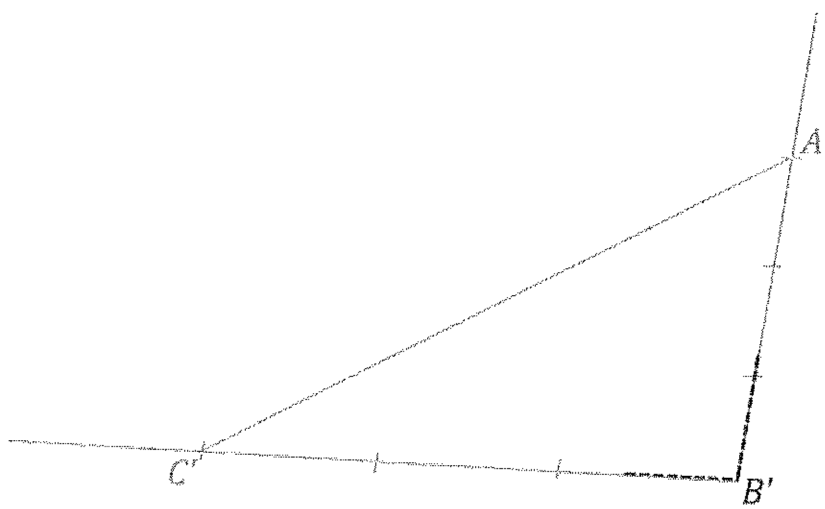
3. Triangle ABC is provided below, and one angle of scale drawing $\triangle A'B'C'$ is also provided. Use construction tools to complete the scale drawing so that the scale factor is $r = 3$. Explain your construction. How do you know that $\triangle A'B'C'$ is in fact a scale drawing of $\triangle ABC$ with scale factor $r = 3$?

Extend both rays from B'. Use the compass to mark off a length equal to $3(AB)$ on one ray and a length equal to $3(BC)$ on the other. Label the ends of the two lengths A' and C', respectively. Join A' to C'.

I should remember that a scale drawing of a figure is made without respect to the orientation of the original figure.

By measurement, each side is three times the length of the corresponding side of the original figure, and all three angles are equal in measurement to the three corresponding angles in the original figure.





Lesson 2: Making Scale Drawings Using the Ratio Method

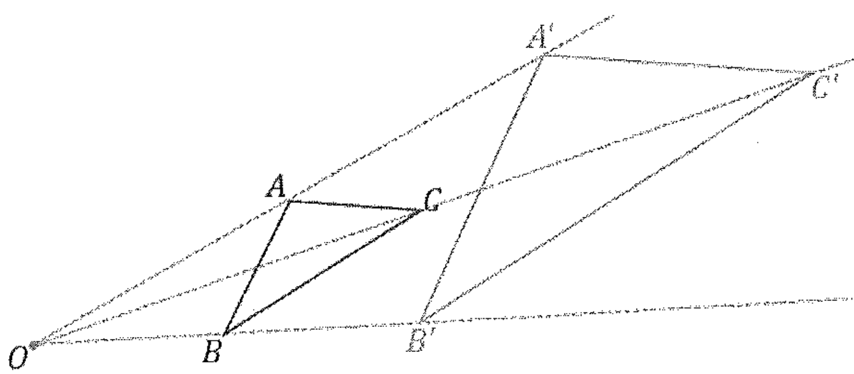
1. What does the ratio method rely on?

   To apply the ratio method, dilate key vertices of the provided figure by the scale factor. The locations of the image points depend on the scale factor and the distance of each point from the center. If the scale factor r is greater than 1, the points will move r times the current distance away from the center; a scale factor r that is less than 1 will mean the vertices move r times the current distance toward the center.

   I should remember that a ruler is critical when applying the ratio method in order to measure and accurately locate image vertices along each ray from the center.

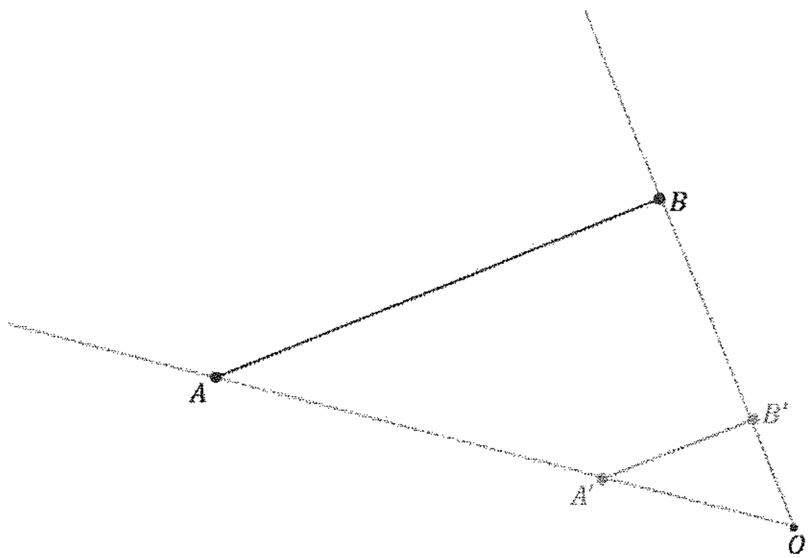
2. Create a scale drawing of the figure below using the ratio method about center O and scale factor $r = 2$.

   I must measure the lengths OA, OB, and OC and double each of those lengths along the respective rays to locate A', B', and C'.



3. Create a scale drawing of the figure below using the ratio method about center O and scale factor $r = \frac{1}{3}$.

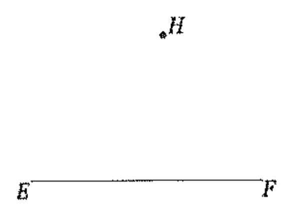
I must measure the lengths OA and OB and take one third of each of those lengths along the respective rays to find A' and B'.



Lesson 3: Making Scale Drawings Using the Parallel Method

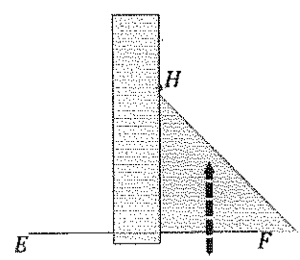
1. Use the step-by-step guide below, a setsquare, and ruler to construct parallelogram $EFGH$, provided side $\overline{EF}$ and vertex H.

Step 1



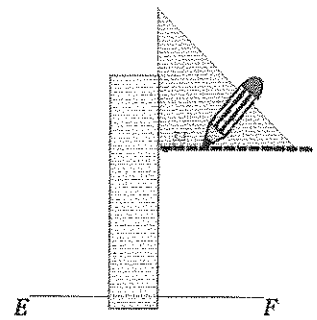
Step 2

Align the setsquare and ruler; one leg of the setsquare should line up with side $\overline{EF}$, and the perpendicular leg should be flush against the ruler.



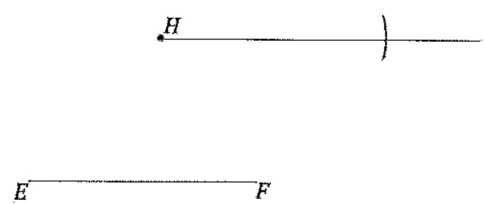
Step 3

Slide the setsquare up until the leg that was lined up with $\overline{EF}$ passes through H. Draw a line that passes through H.



Step 4

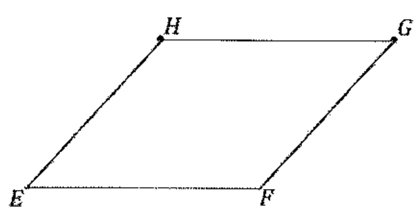
Use a compass to mark off the length of $\overline{EF}$ beginning at H, and label this point as G.



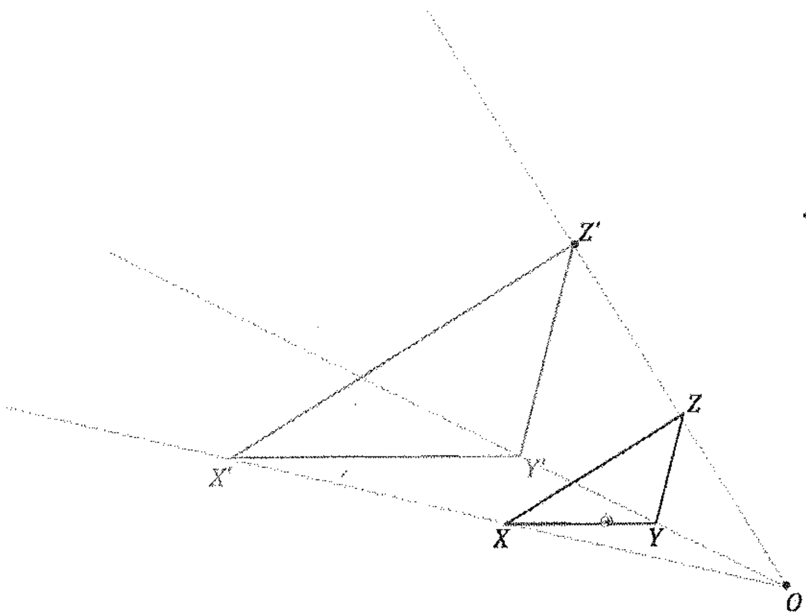
Step 5

Join E to H and F to G.

In order to apply the parallel method, I must be able to use a setsquare and straightedge.



2. Use the parallel method to create a scale drawing of $\triangle XYZ$ about center O, provided Z'.



When using the parallel method, I should remember that one point will be marked for me (e.g., Z'), or I will be provided with a scale factor.

Step 1. Draw a ray beginning at O through each vertex of the figure.

Step 2. Align the setsquare and ruler; one leg of the setsquare should line up with side $\overline{YZ}$, and the perpendicular leg should be flush against the ruler.

Step 3. Slide the setsquare along the ruler until the edge of the setsquare passes through Z'. Then, along the perpendicular leg of the setsquare, draw the segment through Z' that is parallel to $\overline{YZ}$ until it intersects with $\overrightarrow{OY}$, and label this point Y'.

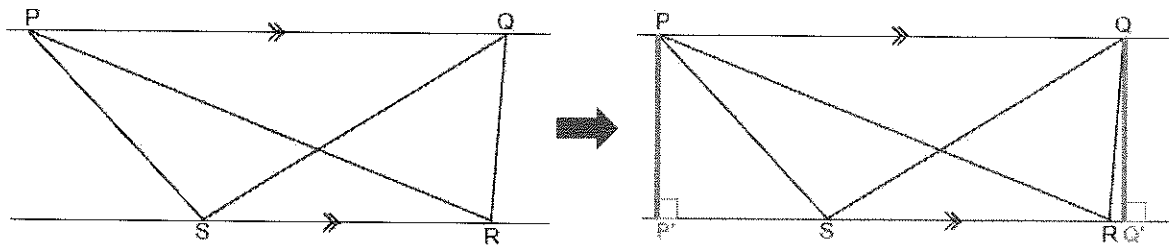
Step 4. Continue to create parallel segments to determine each successive vertex point.

Step 5. Use your ruler to join the final two unconnected vertices (in the case of a triangle, Step 4 is done just once before moving to Step 5, but with a figure with more vertices, Step 4 may be done several times).

Lesson 4: Comparing the Ratio Method with the Parallel Method

1. $\triangle PRS$ and $\triangle QRS$ share the same base $\overline{SR}$ so that points P and Q lie on a line parallel to $\overleftrightarrow{SR}$. Demonstrate why the two triangles have the same area; make additions to the diagram as necessary.

Draw altitude $\overline{PP'}$ for $\triangle PSR$. Draw altitude $\overline{QQ'}$ for $\triangle QSR$.



Quadrilateral $PP'Q'Q$ is a parallelogram. This implies that $PP' = QQ'$. Since the triangles share the same base and have altitudes of equal length, then the areas of the triangles are equal.

$$\text{Area}(\triangle PSR) = \frac{1}{2}SR \cdot PP' = \frac{1}{2}SR \cdot QQ' = \text{Area}(\triangle QSR)$$

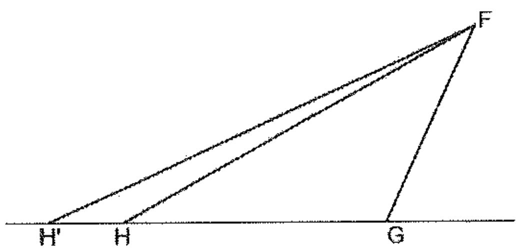
> In order to show that the triangles have the same area, I need to consider the elements needed to find the area such as the height of each triangle.

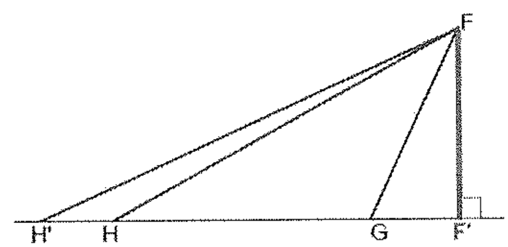
2. In the following diagram, $\overline{GHH'}$. $\triangle FGH$ and $\triangle FGH'$ have different-length bases, $\overline{GH}$ and $\overline{GH'}$, and share vertex F. Demonstrate that the value of the ratio of their areas is equal to the value of the ratio of the lengths of their bases, that is,

By declaring "$\overline{GHH'}$", we know that G, H, and H' are collinear.

$$\frac{\text{Area}(\triangle FGH)}{\text{Area}(\triangle FGH')} = \frac{GH}{GH'}.$$

Make additions to the diagram as necessary.



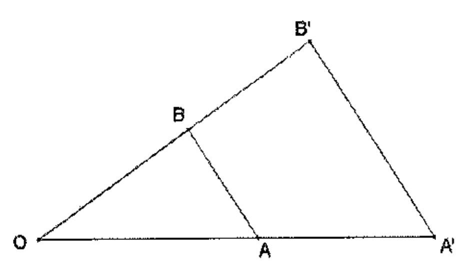


Draw perpendicular $\overleftrightarrow{FF'}$ to $\overleftrightarrow{GH}$; $\overline{FF'}$ is altitude to both $\triangle FGH$ and $\triangle FGH'$.

Use the area formula for triangles:

$$\frac{\text{Area}(\triangle FGH)}{\text{Area}(\triangle FGH')} = \frac{\frac{1}{2}GH \cdot FF'}{\frac{1}{2}GH' \cdot FF'} = \frac{GH}{GH'}.$$

3. In $\triangle OA'B'$, A lies on $\overline{OA'}$, and B lies on $\overline{OB'}$.



   a. If $\overline{AB}$ splits $\overline{OA'}$ and $\overline{OB'}$ proportionally, what is implied?

   $\overline{AB} \parallel \overline{A'B'}$

   b. If $\overline{AB} \parallel \overline{A'B'}$, what is implied? Be specific.

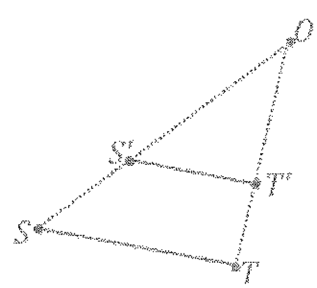
   $\overline{AB}$ is a proportional side splitter, or $\frac{OA'}{OA} = \frac{OB'}{OB}$.

Lesson 5: Scale Factors

1. Point O lies on $\overleftrightarrow{PQ}$. Describe the transformation that occurs based on the diagram below, and state the relationship between pre-image $\overline{PQ}$ and image $\overline{P'Q'}$.

A dilation is applied to $\overline{PQ}$ about O by a scale factor $r > 1$. Since P, O, and Q are collinear, the image points remain on $\overleftrightarrow{PQ}$. By the dilation theorem, $P'Q' = r \cdot PQ$.

2. A dilation is applied to $\overline{ST}$ with a center O that does not lie on $\overleftrightarrow{ST}$. Use a scale factor $r < 1$ to draw a figure that describes this scenario. Use the dilation theorem to describe two facts about $\overline{S'T'}$.



By the dilation theorem, $\overleftrightarrow{ST} \parallel \overleftrightarrow{S'T'}$, and $S'T' = r \cdot ST$.

3. Two different points D and E are dilated from O.

   a. If $DE = 6$, and a scale factor r is 1.5, and O is collinear with D and E, use the dilation theorem to describe two facts that are known about $\overline{D'E'}$.

   $D'E' = 1.5 \cdot 6 = 9$, and $\overleftrightarrow{DE} = \overleftrightarrow{D'E'}$.

   b. If a different center, C, which is not collinear with D and E is used, what changes about the facts known regarding $\overline{D'E'}$?

   $D'E'$ is still 9, but now $\overleftrightarrow{DE} \parallel \overleftrightarrow{D'E'}$.

I must remember that, by the dilation theorem, a segment that does not contain the center of dilation will map to a parallel segment while a segment that contains the center will map to a segment belonging to the same line.

Lesson 6: Dilations as Transformations of the Plane

1. We have studied two categories of transformations. What are they, and what distinguishes each one?

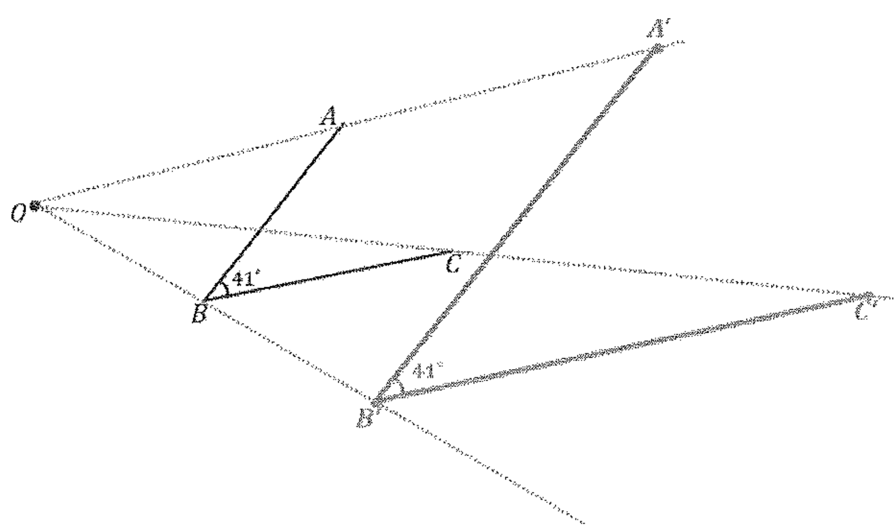
   We have studied transformations that are distance-preserving (rigid) and a kind of transformation that is not distance-preserving (not rigid)—dilations. Distance-preserving transformations—rotations, reflections, and translations—are such that the distance between any two points of a figure is the same as the distance between the images of those two points. Under a dilation from O, a point P is assigned to a point $F(P)$ so that $OP' = r \cdot OP$; this means that distances under a dilation are scaled and are, therefore, not preserved.

2. What property do dilations have in common with basic rigid motions?

   Any transformations map lines to lines, rays to rays, segments to segments, and angles to angles.

   Both rigid motions and dilations preserve angle measure.

3. Use either a ruler or a compass to dilate $\angle ABC$ from O by a scale factor of $r = 2$. Use a protractor to confirm that the dilated angle still has the same angle measure.



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4. Write the inverse dilation that will map the image point P' back to P.

   a. $D_{O,4}(P)$ $\quad$ $D_{O,\frac{1}{4}}\left(D_{O,4}(P)\right) = P$

   b. $D_{O,\frac{3}{4}}(P)$ $\quad$ $D_{O,\frac{4}{3}}\left(D_{O,\frac{3}{4}}(P)\right) = P$

   c. $D_{O,\frac{1}{6}}(P)$ $\quad$ $D_{O,6}\left(D_{O,\frac{1}{6}}(P)\right) = P$

Dilations have inverse functions that return each dilated point back to itself.

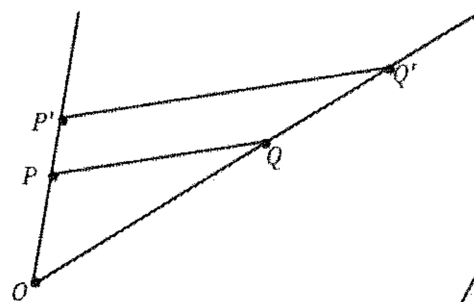
Lesson 7: How Do Dilations Map Segments?

1. Consider a dilation of a segment, $\overline{AB}$, where the center of dilation, O, lies on $\overleftrightarrow{AB}$. Explain why $\overleftrightarrow{AB} = \overleftrightarrow{A'B'}$. Do all the points between A and B map to all the points between A' and B'?

   By definition of dilation, A maps to A' along $\overrightarrow{OA}$, and B maps to B' along $\overrightarrow{OB}$; hence, $\overleftrightarrow{AB} = \overleftrightarrow{A'B'}$. The dilation also maps all of the points between A and B to all of the points between A' and B'.

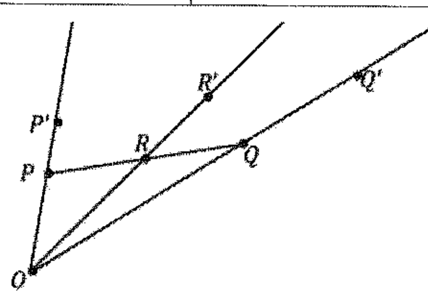
2. Fill in the missing steps of the proof:

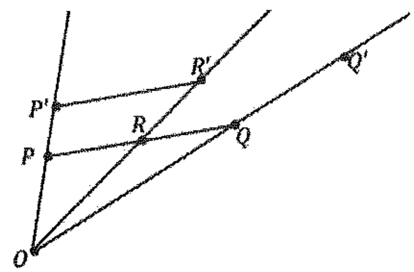
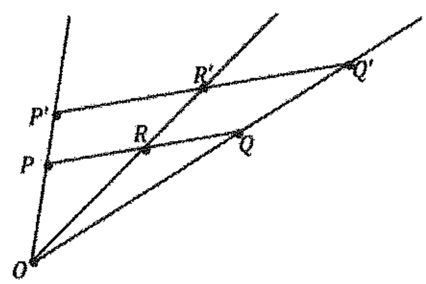
   Let O be a point not on $\overleftrightarrow{PQ}$ and $D_{O,r}$ be a dilation with center O and scale factor $r > 1$ that sends point P to P' and Q to Q'. If R is another point that lies on $\overleftrightarrow{PQ}$, then $D_{O,r}(R)$ is a point that lies on $\overleftrightarrow{P'Q'}$.



My plan is to use the given definition of dilation, triangle side splitter theorem, and parallel postulate to finish the proof.

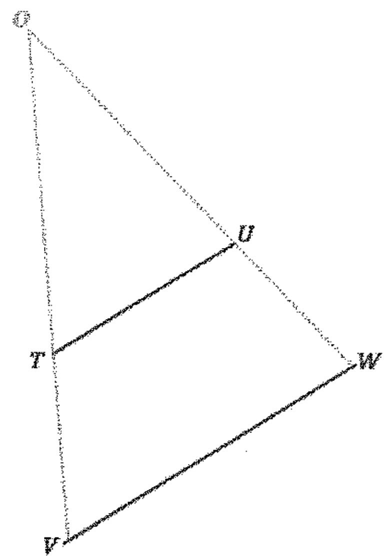
1. A dilation $D_{O,r}$ with center O and scale factor r sends point P to P' and Q to Q'.	Given
2. $\frac{OP'}{OP} = \frac{OQ'}{OQ} = r$	***By definition of dilation***
3. $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$	By the triangle side splitter theorem



4. Let R be a point on segment PQ between P and Q. Let R' be the dilation of R by $D_{O,r}$.	*Given*
5. $\frac{OP'}{OP} = \frac{OR'}{OR} = r$	By definition of dilation
	
6. $\overleftrightarrow{P'R'} \parallel \overleftrightarrow{PQ}$	By the triangle side splitter theorem as applied to $\triangle OPR$ and the fact that $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$ are the same line
	
7. $\overleftrightarrow{P'R'} = \overleftrightarrow{P'Q'}$	***By the parallel postulate which states that through a given external point P', there is at most one line parallel to a given line PQ***
8. ***The point R' lies on $\overleftrightarrow{P'Q'}$.***	By Step 7

3. Parallel segments, $\overline{TU}$ and $\overline{VW}$, in the plane are of different lengths. Will a dilation map $\overline{TU}$ to $\overline{VW}$?

Rays $\overrightarrow{VT}$ and $\overrightarrow{WU}$ intersect at a point, O. Since $\overline{TU} \parallel \overline{VW}$ in $\triangle OVW$, $\overline{TU}$ is a proportional side splitter. If the sides of the triangle are split proportionally, there must be a dilation with the center at O and a scale factor $r = \frac{VO}{TO} = \frac{WO}{UO}$ that maps $\overline{TU}$ to $\overline{VW}$.



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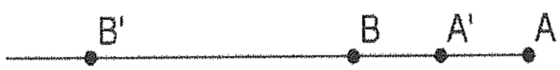
Lesson 8: How Do Dilations Map Lines, Rays, and Circles?

1. In the diagram below, $\overrightarrow{A'B'}$ is the image of $\overrightarrow{AB}$ under a dilation from point O with an unknown scale factor; A maps to A', and B maps to B'. Use direct measurement to determine the scale factor r, and then find the center of dilation O.



> I need to use the definition of dilation and my ruler. I will use centimeters, but I could use inches.

By the definition of dilation, $A'B' = r\,(AB)$, $OA' = r(OA)$, *and* $OB' = r(OB)$.

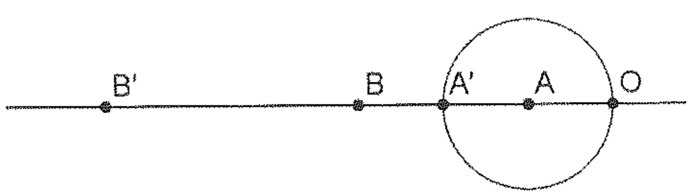
By direct measurement $\frac{A'B'}{AB} = \frac{5}{2.5} = 2 = r$.

The images of A and B are pushed to the left on $\overleftrightarrow{AB}$ under the dilation, and $A'B' > AB$, so the center of dilation must lie on $\overleftrightarrow{AB}$ to the right of points A and B.

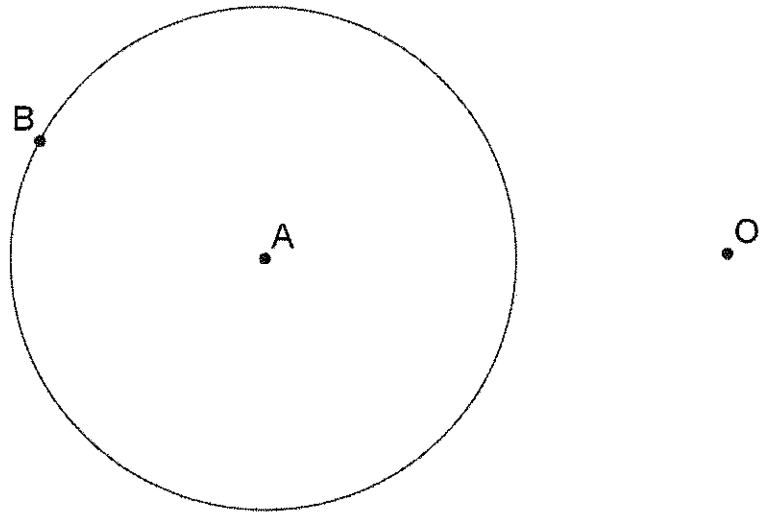
By the definition of dilation,

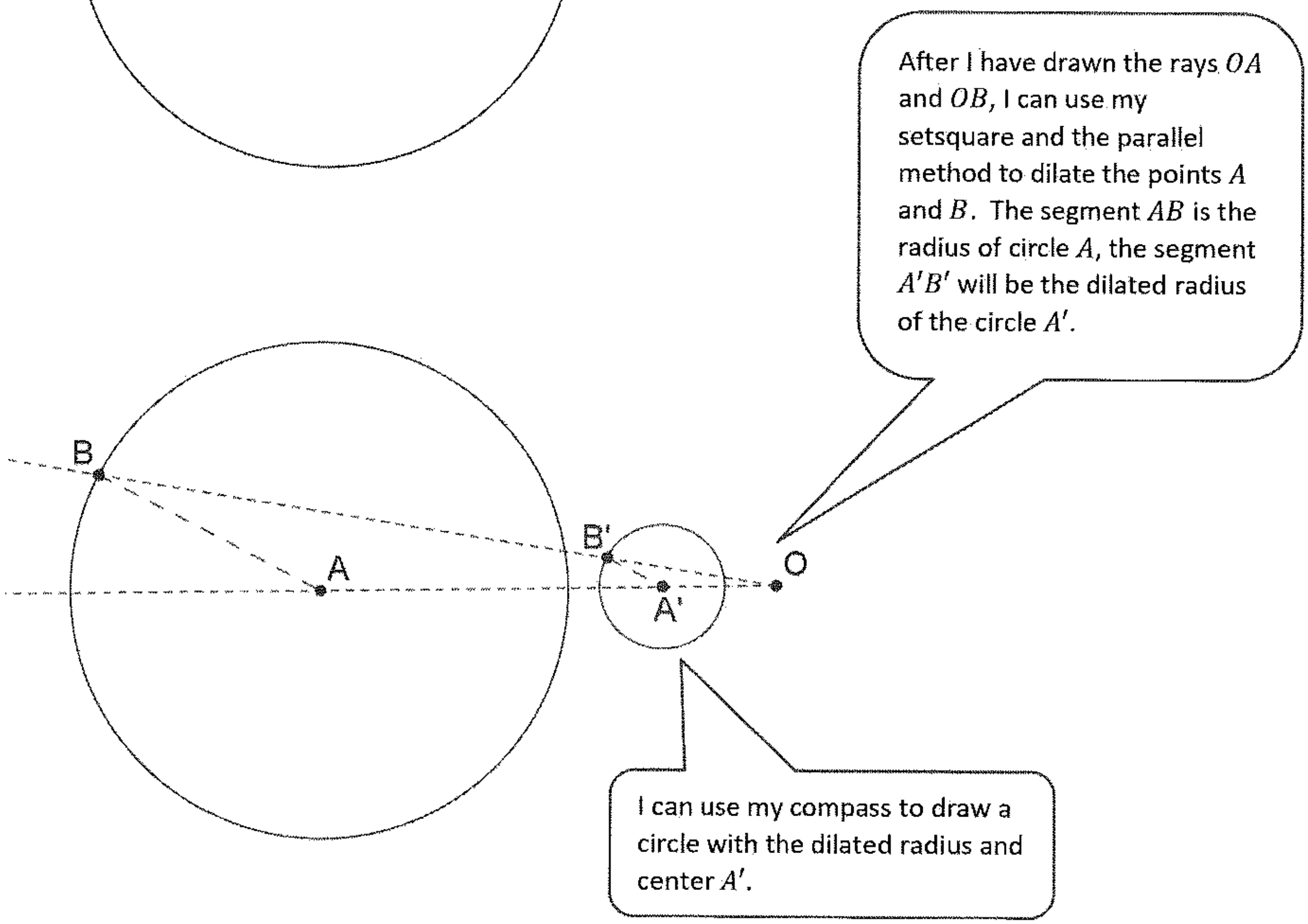
> I remember from the lesson that the length of $\overline{OA'}$ is the union of the lengths of $\overline{OA}$ and $\overline{AA'}$.

$$OA' = 2(OA)$$
$$(OA + AA') = 2(OA)$$
$$\frac{OA + AA'}{OA} = 2$$
$$\frac{OA}{OA} + \frac{AA'}{OA} = 2$$
$$1 + \frac{AA'}{OA} = 2$$
$$\frac{AA'}{OA} = 1$$
$$AA' = 1(OA)$$
$$AA' = OA$$



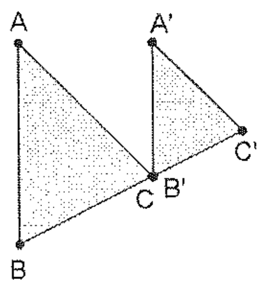
2. Dilate circle A with radius AB from center O with a scale factor $r = \frac{1}{4}$.

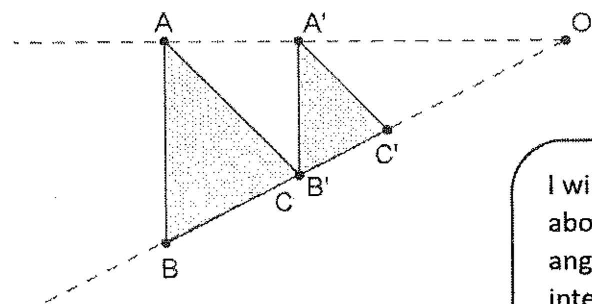




Lesson 9: How Do Dilations Map Angles?

1. Shown below is $\triangle ABC$ and its image $\triangle A'B'C'$ after it has been dilated from center O by scale factor $r = \frac{2}{3}$. Prove that the dilation maps $\triangle ABC$ to $\triangle A'B'C'$ so that $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, and $m\angle C = m\angle C'$.





I will use what I know about corresponding angles and alternate interior angles being equal in measure when parallel lines are cut by a transversal.

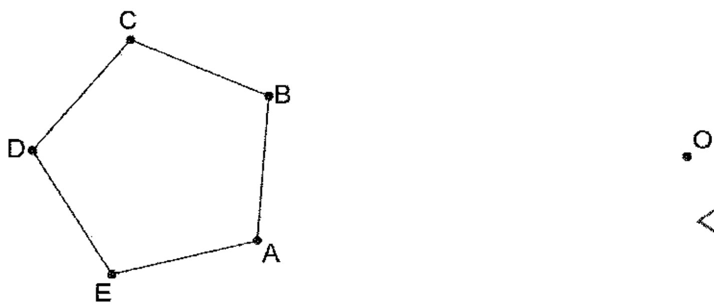
Locate the center of dilation O by drawing rays through each of the pairs of corresponding points. The intersection of the rays is the center of dilation O. Since dilations map segments to segments, and the dilated segments must either coincide with their pre-image or be parallel, then we know that $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$, $\overleftrightarrow{AC} \parallel \overleftrightarrow{A'C'}$, and $\overleftrightarrow{BC} \parallel \overleftrightarrow{B'C'}$.

Then:

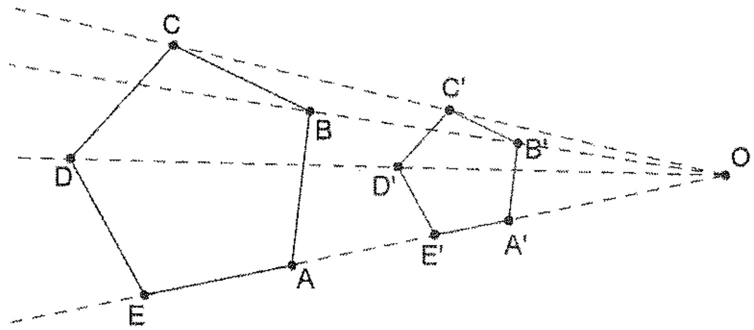
- *$\angle A'B'C'$ is congruent to $\angle ABC$. If parallel lines ($\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$) are cut by a transversal ($\overleftrightarrow{BC}$), then corresponding angles are congruent.*
- *$\angle A'C'B'$ is congruent to $\angle ACB$. If parallel lines ($\overleftrightarrow{AC} \parallel \overleftrightarrow{A'C'}$) are cut by a transversal ($\overleftrightarrow{BC}$), then corresponding angles are congruent.*
- *$\angle A'B'A$ is congruent to $\angle BAC$. If parallel lines ($\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$) are cut by a transversal ($\overleftrightarrow{AB'}$), then alternating interior angles are congruent.*
- *$\angle A'B'A$ is congruent to $\angle B'A'C'$. If parallel lines ($\overleftrightarrow{AC} \parallel \overleftrightarrow{A'C'}$) are cut by a transversal ($\overleftrightarrow{A'B'}$), then alternate interior angles are congruent.*

By the transitive property, $\angle BAC \cong \angle A'B'A \cong \angle B'A'C'$ and $\angle B'A'C' \cong \angle BAC$. Since congruent angles are equal in measure, $m\angle B'A'C' = m\angle A'$, and $m\angle BAC = m\angle A$; then, $m\angle A = m\angle A'$. Similar reasoning shows that $m\angle B = m\angle B'$, and $m\angle C = m\angle C'$.

2. Dilate pentagon $ABCDE$ from center O using a scale factor of $r = \frac{1}{2}$.

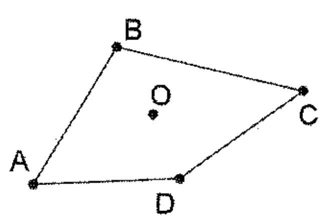


I know the ratio of the corresponding sides will be the same and the corresponding interior angles will be the same measure in the dilated pentagon.

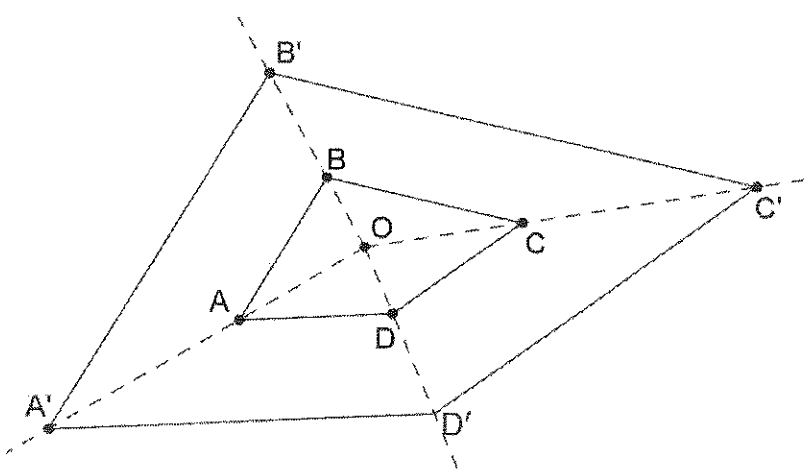


A dilation maps a pentagon to a pentagon so that the ratio of corresponding sides is the same and corresponding interior angles are the same measure.

3. Dilate quadrilateral $ABCD$ from center O using a scale factor $r = 2\frac{1}{2}$.



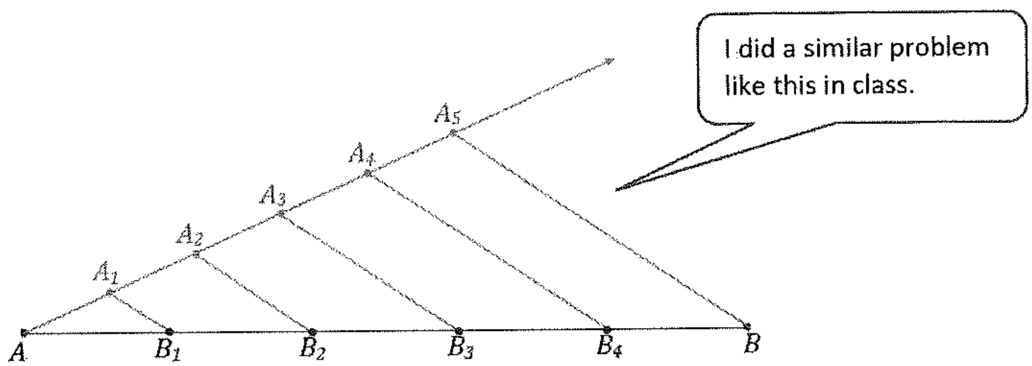
A dilation maps a quadrilateral to a quadrilateral so that the ratio of corresponding sides is the same and corresponding interior angles are the same measure.



Since the center of dilation is inscribed in the quadrilateral, and the scale factor is $r > 1$, I know the dilated quadrilateral will circumscribe the pre-image.

Lesson 10: Dividing the King's Foot into 12 Equal Pieces

1. In the following diagram, $A_1, A_2, \ldots, A_5$ have been constructed to be equally spaced, and segments $A_1B_1, A_2B_2, \ldots, A_5B$ are parallel by construction. Is segment AB divided into five equal lengths? Justify your answer.

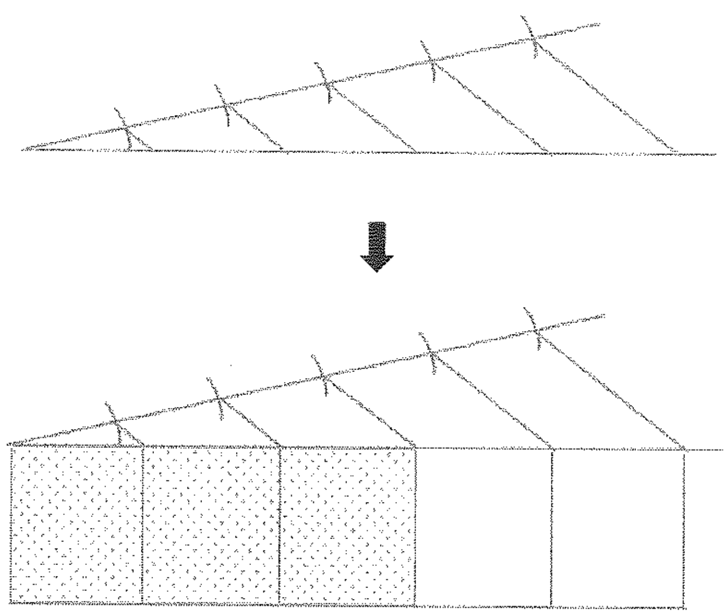


Segments A_1B_1 and A_2B_2 are parallel by construction. By the triangle side splitter theorem, both segments are proportional side splitters of triangle ABA_5. We also know that $A_1, A_2, \ldots, A_5$ are equally spaced by construction. This means that $\frac{AA_1}{AA_5} = \frac{AB_1}{AB} = \frac{1}{5}$ and that $\frac{AA_2}{AA_5} = \frac{AB_2}{AB} = \frac{2}{5}$. By similar reasoning, we can show that $\frac{AB_3}{AB} = \frac{3}{5}$, and $\frac{AB_4}{AB} = \frac{4}{5}$. Then, $AB_1 = \frac{1}{5}AB$, $AB_2 = \frac{2}{5}AB$, $AB_3 = \frac{3}{5}AB$, and $AB_4 = \frac{4}{5}AB$. Then, segment AB is divided into five equal lengths.

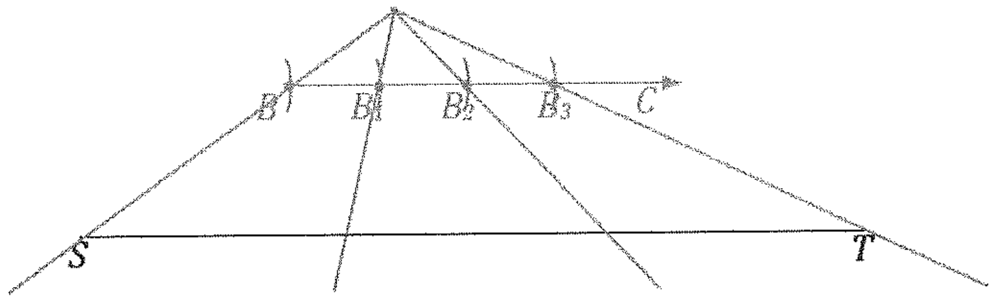
2. Draw a rectangle that is exactly $\frac{3}{5}$ shaded.

One solution might be to first use the side splitter method to divide a segment into five equal lengths and then build a rectangle around the endpoints of the divided segment.

Possible solution:



3. Use the dilation method to divide $\overline{ST}$ into three equal lengths.

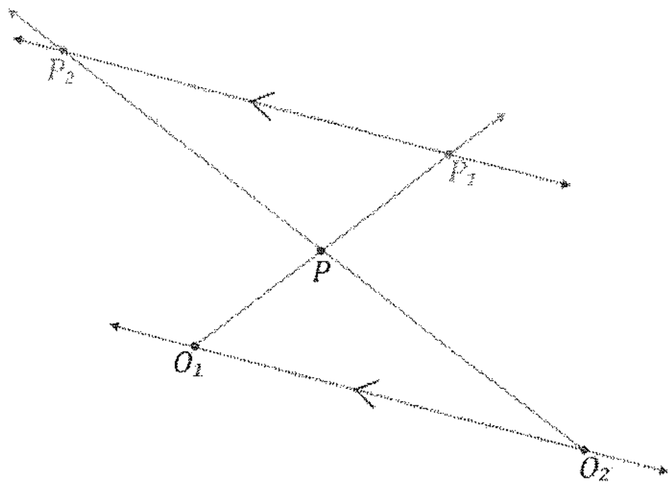


When using the dilation method, I remember to make the ray parallel to the segment clearly longer or shorter than the segment.

Lesson 11: Dilations from Different Centers

1. Dilate the point P from centers O_1 and O_2 by a scale factor of $r = 2$. What do you notice about the line that passes through P_1 and P_2 with respect to the line that passes through O_1 and O_2?

The dilation from two different centers with the same scale factor will result in the line passing through the two centers and the line passing through the two dilated points being parallel.



Lines P_1P_2 and O_1O_2 are parallel.

2. $\triangle A_1B_1C_1$ is a scale drawing of $\triangle ABC$ with scale factor r_1, and $\triangle A_2B_2C_2$ is a scale drawing of $\triangle A_1B_1C_1$ with scale factor r_2.

   a. Determine the scale factor r_1 for $\triangle A_1B_1C_1$ (relative to $\triangle ABC$).

   I can measure and use the definition of dilation to determine the scale factors.

   $r_1 = 3$

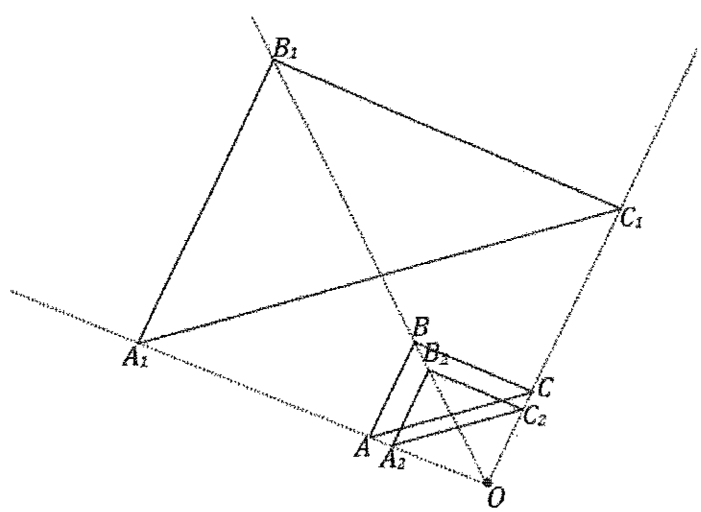
   b. Determine the scale factor r_2 for $\triangle A_2B_2C_2$ (relative to $\triangle A_1B_1C_1$).

   $r_2 = \frac{1}{4}$

   c. Show the calculation to find the value of the scale factor, r, of the dilation that takes $\triangle ABC$ to $\triangle A_2B_2C_2$.

   The scale factor that takes $\triangle ABC$ to $\triangle A_2B_2C_2$ is a product of the individual scale factors of the composition of dilations.

   $r = r_1r_2 = \frac{3}{4}$



3. Segment A_1B_1 is dilated by a scale factor r_1 from O_1 and results in segment A_2B_2; segment A_2B_2 is dilated by a scale factor r_2 from O_2 and results in segment A_3B_3.

   a. Determine the scale factor r_1 of the dilation that results in scale drawing A_2B_2.

   $r_1 = \frac{1}{2}$

   b. Determine the scale factor r_2 of the dilation that results in scale drawing A_3B_3.

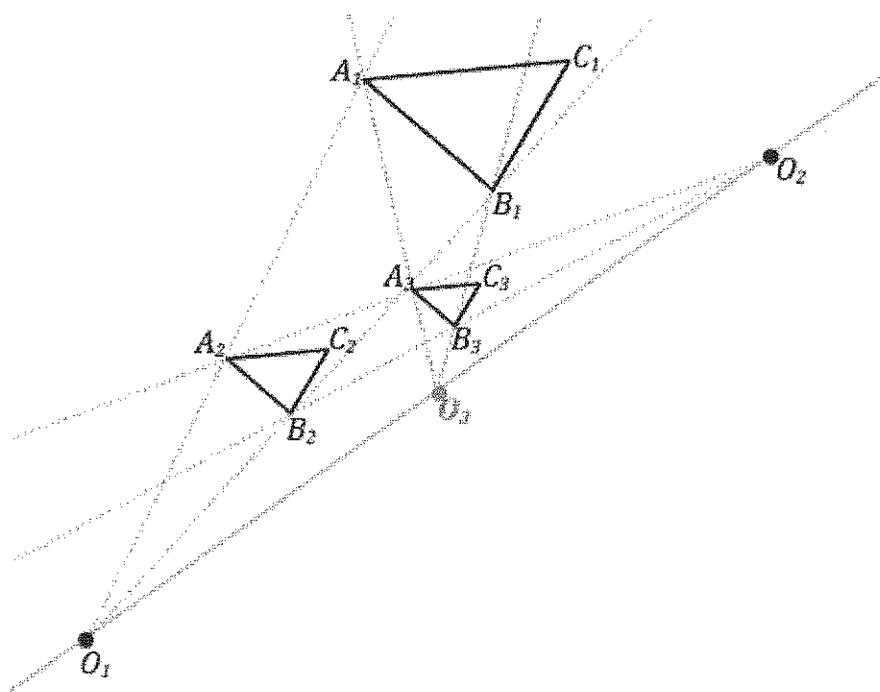
   $r_1 = \frac{2}{3}$

   c. Determine the center of the dilation, O_3, that would map $\overline{A_1B_1}$ to $\overline{A_3B_3}$. What is the value of the scale factor, r, of this dilation?

   $r = r_1r_2 = \frac{1}{3}$

   d. What is significant about the position of O_3 with respect to O_1 and O_2?

   O_3 is collinear with O_1 and O_2.



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Lesson 12: What Are Similarity Transformations, and Why Do We Need Them?

1. What is a similarity transformation?

   A similarity transformation is a composition of a finite number of dilations or basic rigid motions. The scale factor of a similarity transformation is the product of the scale factors of the dilations in the composition. If there are no dilations in the composition, the scale factor is defined to be 1.

   A similarity transformation does not need to have a dilation. A square is a special type of rectangle, but the relationship does not work in reverse; a similar such relationship exists between congruence transformations and similarity transformations. A congruence transformation is a similarity transformation where the scale factor is 1. Similarity transformations generalize the notion of congruency.

2. Figure P is similar to figure P'. Describe a similarity transformation that maps P onto P' with as much detail as possible.

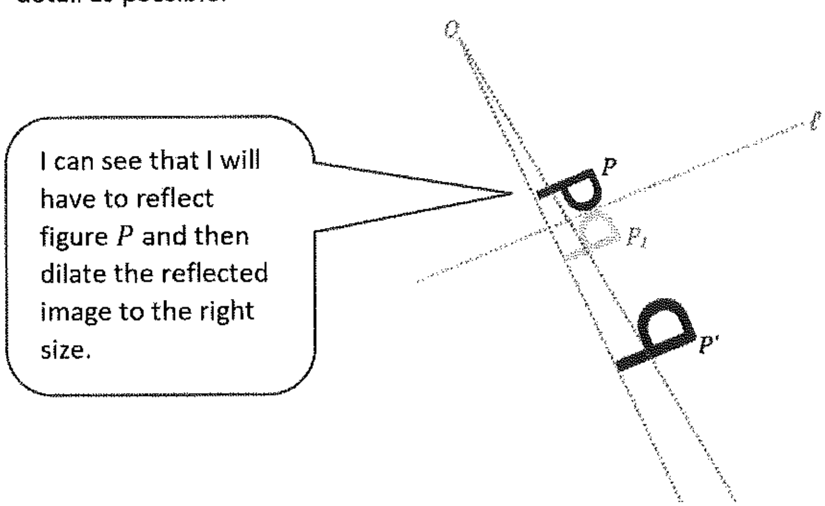


Figure P is reflected over a line ℓ, yielding P_1, and then dilated from a center, O, by a scale factor of $r > 1$, resulting in P'.

3. Figure F is similar to figure F'. Describe a similarity transformation that maps F onto F' with as much detail as possible.

I will have to use basic rigid motions and dilations to describe the similarity transformation.

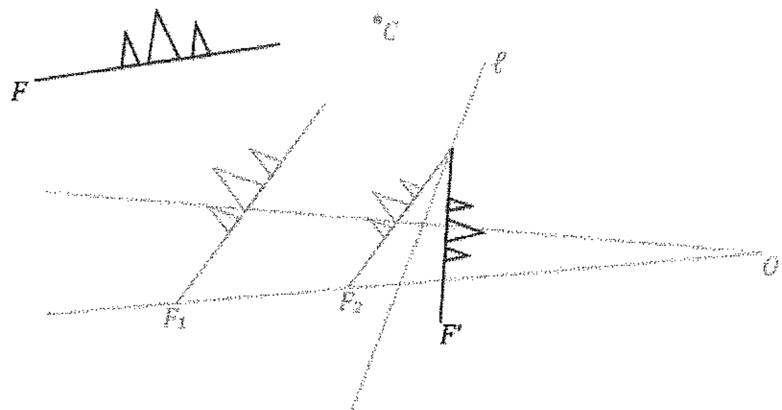
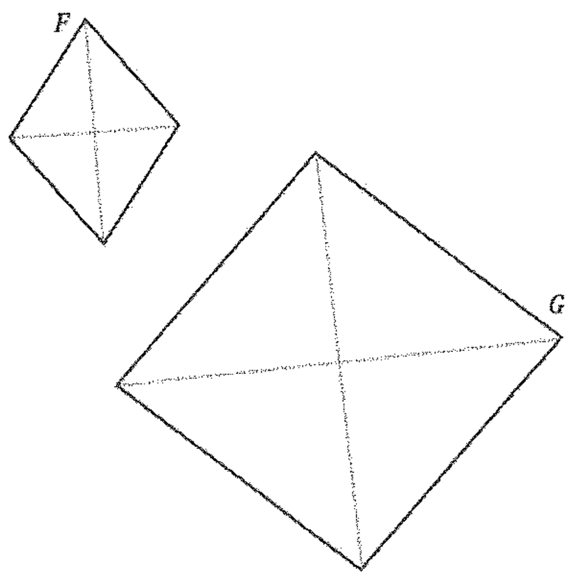


Figure F is rotated about C, yielding the image F_1. Then F_1 is dilated from a center, O, by a scale factor of $r < 1$, resulting in F_2. Finally, F_2 is reflected over a line ℓ, resulting as the figure F'.

4. Is there a similarity transformation that maps rhombus F to rhombus G? Use measurements to justify your answer.



There is no similarity transformation that maps figure F to figure G since any dilation involved in such a transformation would create a scale drawing of figure F, which G is not. This can be proven by comparing corresponding sides to corresponding diagonals, which are not in the same proportion.

Lesson 13: Properties of Similarity Transformations

Use a compass, protractor, and straightedge for the following problems.

1. Describe the properties shared by transformations.

   1) Distinct points are mapped to distinct points.

   2) Each point P' in the plane has a pre-image.

   3) There is a scale factor r for G so that for any pair of points P and Q with images $P' = G(P)$ and $Q' = G(Q)$, then $P'Q' = rPQ$.

   4) A similarity transformation sends lines to lines, rays to rays, line segments to line segments, and parallel lines to parallel lines.

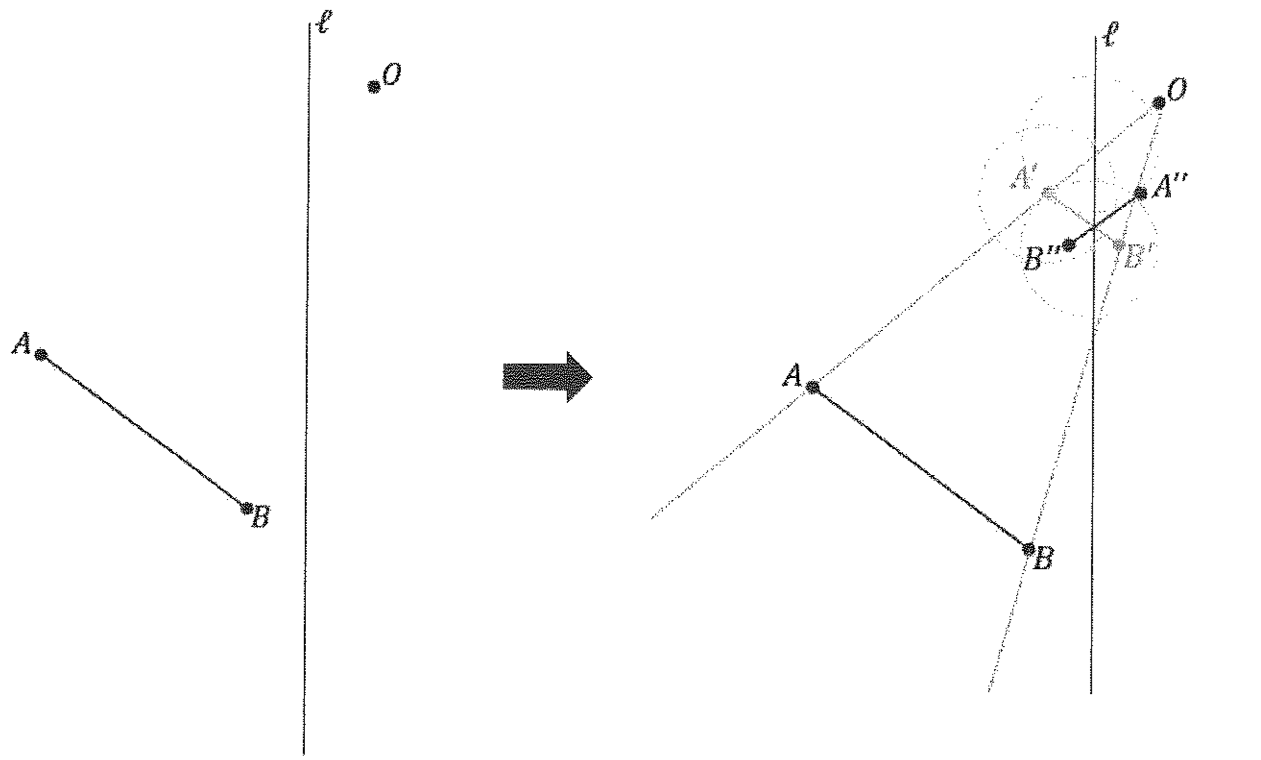
   5) A similarity transformation sends angles to angles of equal measure.

   6) A similarity transformation maps a circle of radius R to a circle of radius rR, where r is the scale factor of the similarity transformation.

   7) All of the properties are satisfied by a similarity transformation consisting of a single translation, reflection, rotation, or dilation. If the similarity transformation consists of more than one such transformation, then the properties still hold because they hold one step at a time.

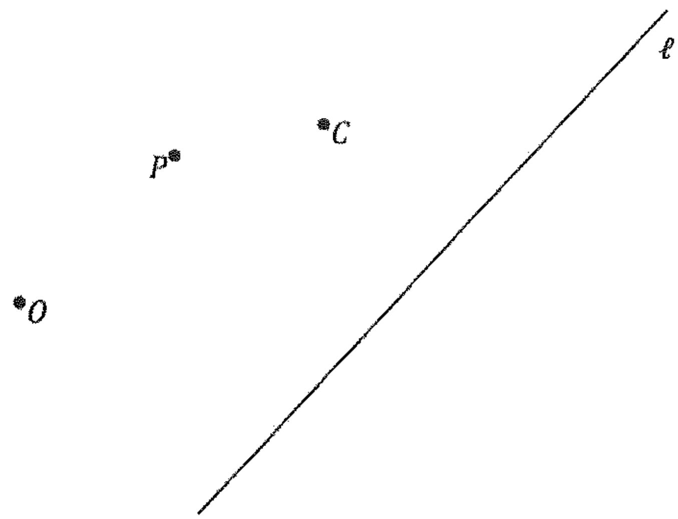
I must remember that these properties are true for dilations, rotations, reflections, and translations. Some properties, like property 3, may seem to apply only to dilations, but property 3 is applicable for any transformation because in a congruence, for example, the scale factor is 1, which makes the property true.

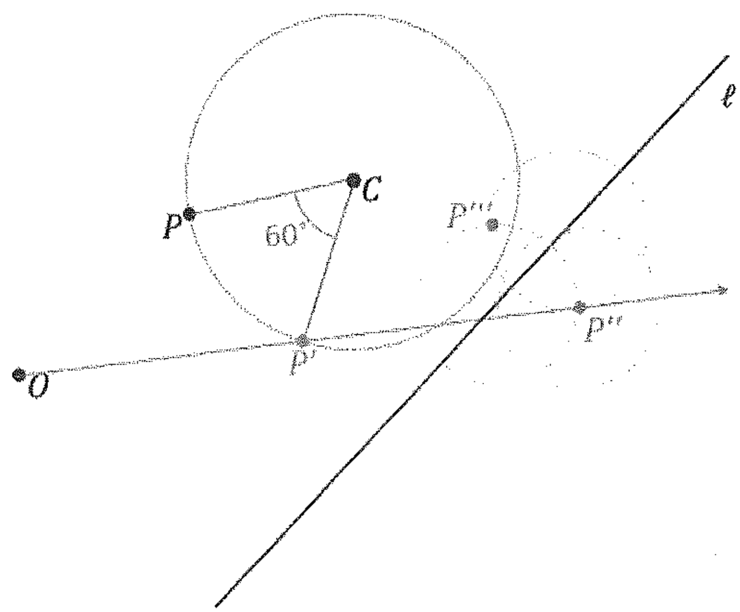
2. The image of segment AB under a similarity transformation is segment $A''B''$. Apply the similarity transformation if the composition is a dilation from O with scale factor $r = \frac{1}{3}$ followed by a reflection over line ℓ.



3. Locate point P''' by applying the following similarity transformation: $r_\ell\left(D_{O,2}\left(R_{C,60^\circ}(P)\right)\right)$.

The order in which I apply the similarity transformation is rotation, dilation, and then reflection.

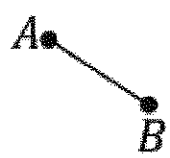




Lesson 14: Similarity

Use a compass, protractor, and straightedge for the following problems.

1. Similarity is reflexive. What does this mean? Sketch or describe two examples that support your answer.



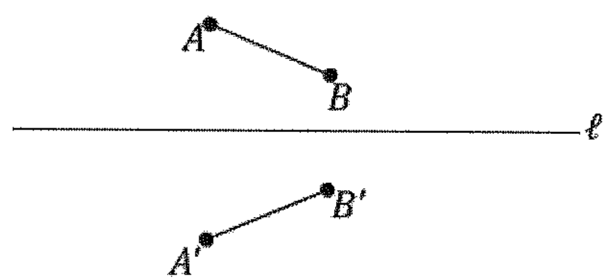
We learned that similarity is reflexive. That means that a figure in a plane can be mapped to itself by a similarly transformation.

Possible solutions:

Example 1

$\overline{AB} = r_\ell(r_\ell(\overline{AB}))$

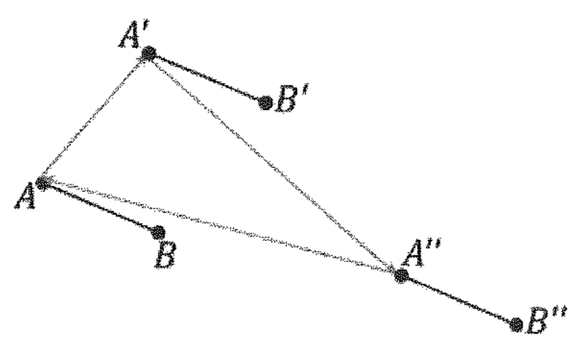
A sequence of two reflections over line ℓ will map $\overline{AB}$ to itself.



Example 2

$\overline{AB} = T_{\overrightarrow{A''A}}\left(T_{\overrightarrow{A'A''}}\left(T_{\overrightarrow{AA'}}(\overline{AB})\right)\right)$

A sequence of three translations maps $\overline{AB}$ back onto itself.



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2. Similarity is symmetric. What does this mean? Describe an example that supports your answer.

When we say similarity is symmetric, we mean if a figure A in the plane is similar to a figure B, then it must be true that B is similar to A or that there exists a similarity transformation that will map B to A.

This is true because for every composition of transformations, there is a composition that will undo or reverse the first composition. For example, a composition of a dilation by a scale factor of 3 followed by a rotation of $30°$ can be undone by a rotation of $-30°$ followed by a dilation by a scale factor of $\frac{1}{3}$.

Once a similarity transformation is determined to take a figure to another, there is an inverse transformation that will take the figure back to the original.

3. Similarity is transitive. What does this mean? Describe an example that supports your answer.

When we say similarity is transitive, we mean if a figure A in the plane is similar to a figure B, and if B is similar to a third figure C, then it must be true that A is similar to C or that there exists a similarity transformation that will map A to C.

This is true because if there is a composition of transformations T_1 that maps A to B and a composition of transformations T_2 that maps B to C, then the composition of T_1 and T_2 together will map A to C.

For example, if T_1 is a composition of a reflection across a line ℓ, followed by a rotation of $90°$ about a center C and T_2 is a composition of a translation by vector $\overrightarrow{AB}$ followed by a dilation from O by a scale factor of 2, then the similarity transformation that will map A to C is

$$C = D_{O,2}\left(T_{\overrightarrow{AB}}\left(R_{C,90°}\left(r_{\ell}(A)\right)\right)\right).$$

4. A correspondence exists between $\triangle EFG$ and $\triangle XYZ$ so that $\triangle EFG \leftrightarrow \triangle XYZ$. Under this correspondence, $m\angle E = m\angle X$, $m\angle F = m\angle Y$, $m\angle G = m\angle Z$, and $\frac{XY}{EF} = \frac{YZ}{FG} = \frac{ZX}{GE}$. Demonstrate why $\triangle EFG \sim \triangle XYZ$.

I will use my definition of similarity to show why $\triangle EFG \sim \triangle XYZ$.

Since corresponding side lengths of similar triangles are proportional, a dilation exists so that the scale factor is $r = \frac{XY}{EF} = \frac{YZ}{FG} = \frac{ZX}{GE}$. This means that the dilation with scale factor r maps $\triangle EFG$ to $\triangle E'F'G'$ so that $E'F' = rEF$ or $E'F' = \frac{XY}{EF}EF$, $F'G' = \frac{YZ}{FG}FG$, and $G'E' = \frac{ZX}{GE}GE$. This means that $\triangle E'F'G' \cong \triangle XYZ$. Since the triangles are congruent, there must exist a congruence that maps $\triangle E'F'G'$ to $\triangle XYZ$. Then together, the dilation and the congruence imply that a similarity transformation exists that maps $\triangle EFG$ to $\triangle XYZ$, and so $\triangle EFG \sim \triangle XYZ$.

Similarity is symmetric because once a similarity transformation is determined to take a figure to another, there are inverse transformations that will take the figure back to the original.

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Lesson 15: The Angle-Angle (AA) Criterion for Two Triangles to Be Similar

Use a compass, protractor, and straightedge for the following problems.

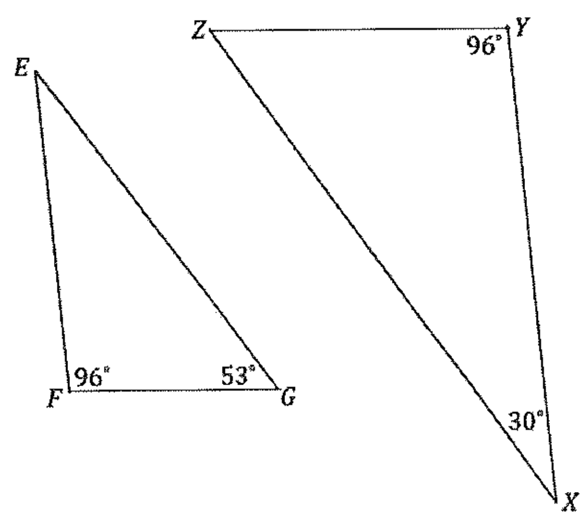
1. State the angle-angle (AA) criterion for similarity. Why is it necessary to establish that only two pairs of angles are equal in measure?

   Two triangles with two pairs of corresponding angles of equal measure are similar. If two corresponding pairs of angles of two triangles are equal in measure, then, by the triangle sum theorem, the third pair of corresponding angles must also be equal in measure. Therefore, it is only necessary to establish that two pairs of angles are equal in measure.

2. Are the following triangles similar?

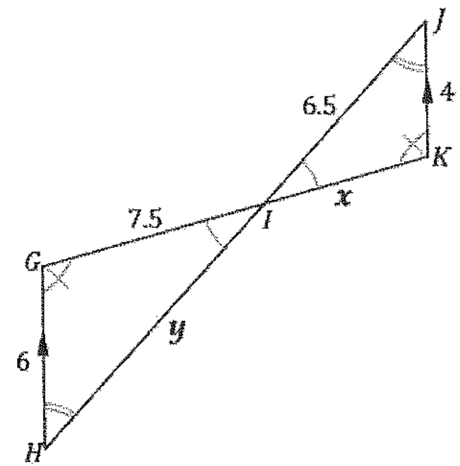
   I can use my answer from Problem 1 to help me with this problem.

   The triangles are not similar. They each have one angle with a measure of $96°$, but the remaining two angles are not equal in measure between the two triangles. By the triangle sum theorem, the remaining angle measures of $\triangle EFG$ are $53°$ and $31°$, and the remaining angle measures of $\triangle XYZ$ are $54°$ and $30°$.



3. Are the following triangles similar? If so, explain why, and solve for the values of x and y.

> My plan is to use the AA criterion. I will use what I know about angles formed when parallel lines are cut by a transversal and vertical angles.



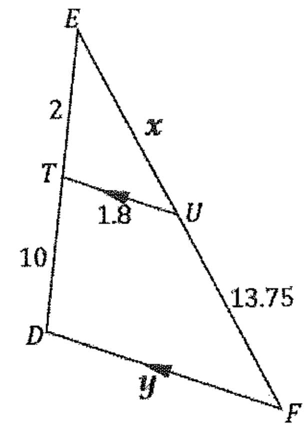
Vertical angles $\angle GIH$ and $\angle KIJ$ are equal in measure, and since $\overline{GH} \parallel \overline{KJ}$, then $m\angle IGH = m\angle IKJ$, and $m\angle IJK = m\angle IHG$ because if parallel lines are cut by a transversal, alternate interior angles are equal in measure. Then, by the AA criterion, the triangles are similar.

Since the triangles are similar, I can now solve for the values of x and y.

$$\frac{y}{6.5} = \frac{6}{4}$$
$$y = 9.75$$

$$\frac{7.5}{x} = \frac{6}{4}$$
$$x = 5$$

4. Is $\triangle ETU$ similar to $\triangle EDF$? Explain. If the triangles are similar, find the values of x and y.



$\triangle ETU$ is similar to $\triangle EDF$ by the AA criterion for similarity. Both triangles share $\angle E$, and since $\overline{TU} \parallel \overline{DF}$, $m\angle ETU = m\angle EDF$, and $m\angle EUT = m\angle EFD$ (if parallel lines are cut by a transversal, then corresponding angles are equal in measure).

$$\frac{2}{12} = \frac{x}{(x + 13.75)}$$

$$2x + 27.5 = 12x$$

$$27.5 = 10x$$

$$x = 2.75$$

$$\frac{2}{12} = \frac{1.8}{y}$$

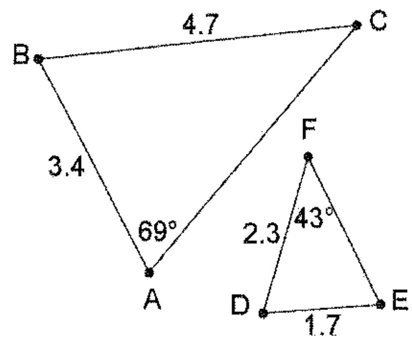
$$2y = 21.6$$

$$y = 10.8$$

Lesson 16: Between-Figure and Within-Figure Ratios

1. $\triangle DEF \sim \triangle ABC$ All side length measurements are in centimeters. Use between-figure ratios to determine the unknown side lengths.

Using the given similarity statement, $\angle D$ corresponds with $\angle A$, and $\angle C$ corresponds with $\angle F$, so it follows that $\overline{AB}$ corresponds with $\overline{DE}$, $\overline{AC}$ with $\overline{DF}$, and $\overline{BC}$ with $\overline{EF}$.



I know that with between-figure ratios, one number comes from one triangle, and the other number in the ratio comes from a different similar triangle.

$$\frac{AB}{DE} = \frac{BC}{EF} \qquad \frac{AB}{DE} = \frac{AC}{DF}$$

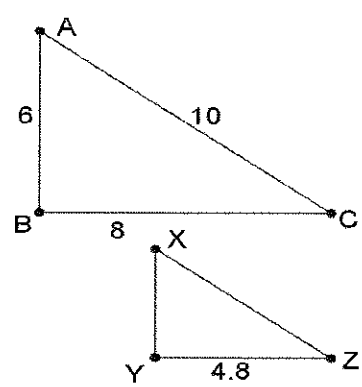
$$\frac{3.4}{1.7} = \frac{4.7}{EF} \qquad \frac{3.4}{1.7} = \frac{AC}{2.3}$$

$$3.4(EF) = 7.99 \qquad 1.7(AC) = 7.82$$

$$EF = 2.35 \qquad AC = 4.6$$

The lengths of $\overline{EF}$ and $\overline{AC}$ are 2.35 centimeters and 4.6 centimeters, respectively.

2. Given $\triangle ABC \sim \triangle XYZ$, answer the following questions:



   a. Write and find the value of the ratio that compares the height $\overline{AB}$ to the hypotenuse of $\triangle ABC$.

   $$\frac{6}{10} = \frac{3}{5}$$

   b. Write and find the value of the ratio that compares the base $\overline{BC}$ to the hypotenuse of $\triangle ABC$.

   $$\frac{8}{10} = \frac{4}{5}$$

   c. Write and find the value of the ratio that compares the height $\overline{AB}$ to the base $\overline{BC}$ of $\triangle ABC$.

   $$\frac{6}{8} = \frac{3}{4}$$

d. Use within-figure ratios to find the corresponding height of $\triangle XYZ$.

$$\frac{AB}{BC} = \frac{XY}{YZ}$$
$$\frac{3}{4} = \frac{XY}{4.8}$$
$$4(XY) = 14.4$$
$$XY = 3.6$$

I know that for within-figure ratios, one ratio contains numbers that represent side lengths from one triangle, and the second ratio contains numbers that represent the side lengths from a second similar triangle.

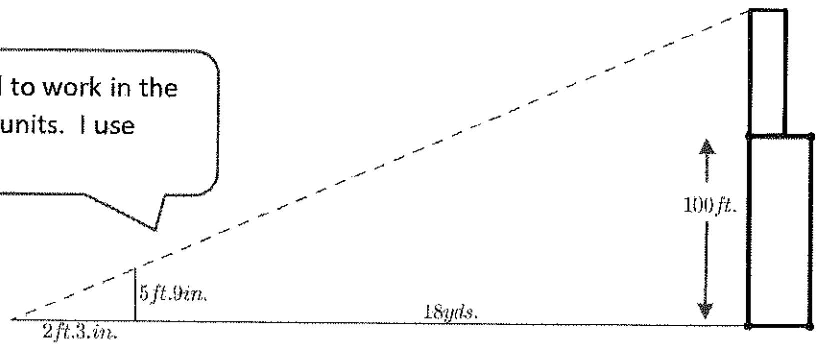
e. Use within-figure ratios to find the hypotenuse of $\triangle XYZ$.

$$\frac{AB}{AC} = \frac{XY}{XZ}$$
$$\frac{3}{5} = \frac{3.6}{XZ}$$
$$3(XZ) = 18$$
$$XZ = 6$$

I can use my answer from part (d) to determine the hypotenuse of $\triangle XYZ$.

3. Pete is wondering how tall the big screen is that rests on a 100-foot tall platform. Pete asks his friend Marci, who is 5 ft. 9 in. tall, to stand approximately 18 yards from the base of the platform. Lying on the ground, Pete visually aligns the top of Marci's head with the top of the big screen and marks his location on the ground approximately 2 ft. 3 in. from his friend. Use Pete's measurements to approximate the height of the big screen.

I need to work in the same units. I use feet.



I know the two nested triangles formed are similar by AA criterion.

Pete's location on the ground is approximately 56.25 ft. ***from the base of the platform. His visual line forms two similar right triangles with the height of the big screen and platform and the height of his friend. Let*** x ***represent the height of the big screen.***

I can use between-figure ratios or within-figure ratios.

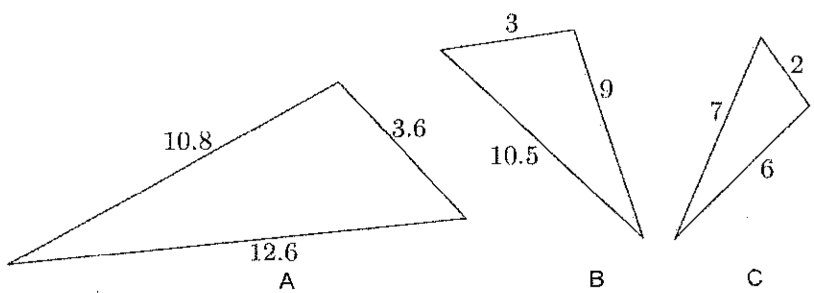
$$\frac{56.25}{2.25} = \frac{x + 100}{5.75}$$
$$323.4375 = 2.25x + 225$$
$$98.4375 = 2.25x$$
$$43.75 = x$$

The height of the larger triangle is 100 added to the height of the big screen.

The height of the big screen is about 43.75 ft. ***(or*** 43 ft. 9 in.***).***

Lesson 17: The Side-Angle-Side (SAS) and Side-Side-Side (SSS) Criteria for Two Triangles to Be Similar

1. State which of the three triangles, if any, are similar and why.



I need to check the corresponding sides of all the triangles (i.e., A and B, A and C, and B and C) to determine if I can use the SSS theorem.

Triangles A and C are similar because they share three pairs of corresponding sides that are in the same ratio.

$$\frac{3.6}{2}=\frac{10.8}{6}=\frac{12.6}{7}=\frac{9}{5}$$

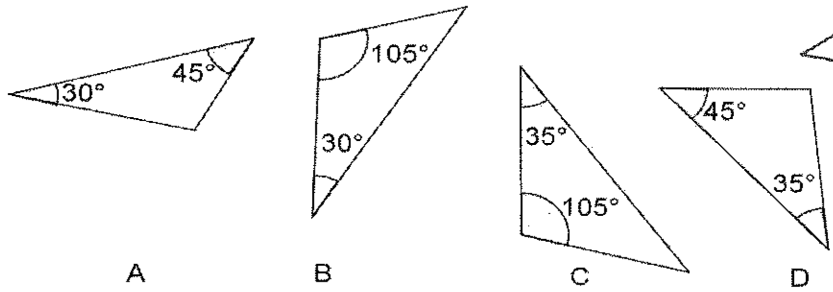
Triangles A and B are similar because the ratios of their corresponding sides are in the same ratio.

$$\frac{3.6}{3}=\frac{10.8}{9}=\frac{12.6}{10.5}=\frac{6}{5}$$

This is true by the transitive property of similarity.

Further, if triangle A is similar to triangle C, and triangle A is similar to B, then triangle B is similar to triangle C.

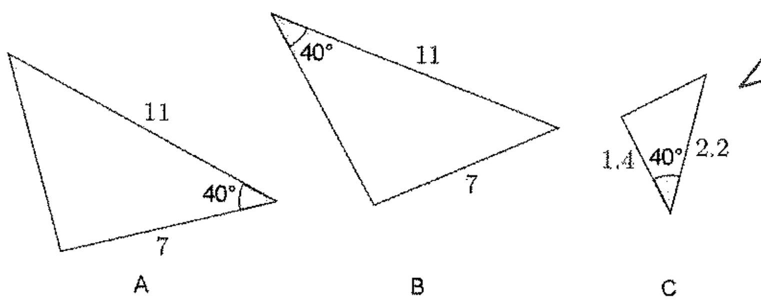
2. State which of the four triangles, if any, are similar and why.



I will use the angle sum of a triangle to determine the missing angle in each triangle. I will use the AA criterion for similarity where applicable.

Triangles A and B are the only similar triangles because they have the same angle measures. Using the angle sum of a triangle, both triangles A and B have angles of $30°$, $45°$, and $105°$.

3. State which of the three triangles, if any, are similar and why.



If I have two known sides, the angle must be the included angle. That means the ratio of the corresponding adjacent sides of the congruent angle must be the same to use the SAS theorem.

Triangles A and C are similar because they have two pairs of corresponding sides that are in the same ratio, and their included angles are equal measures. Triangle B cannot be shown to be similar because, even though it has two sides that are the same length as two sides of triangle A, the $40°$ angle in triangle B is not the included angle and, therefore, does not correspond to the $40°$ angle in triangle A.

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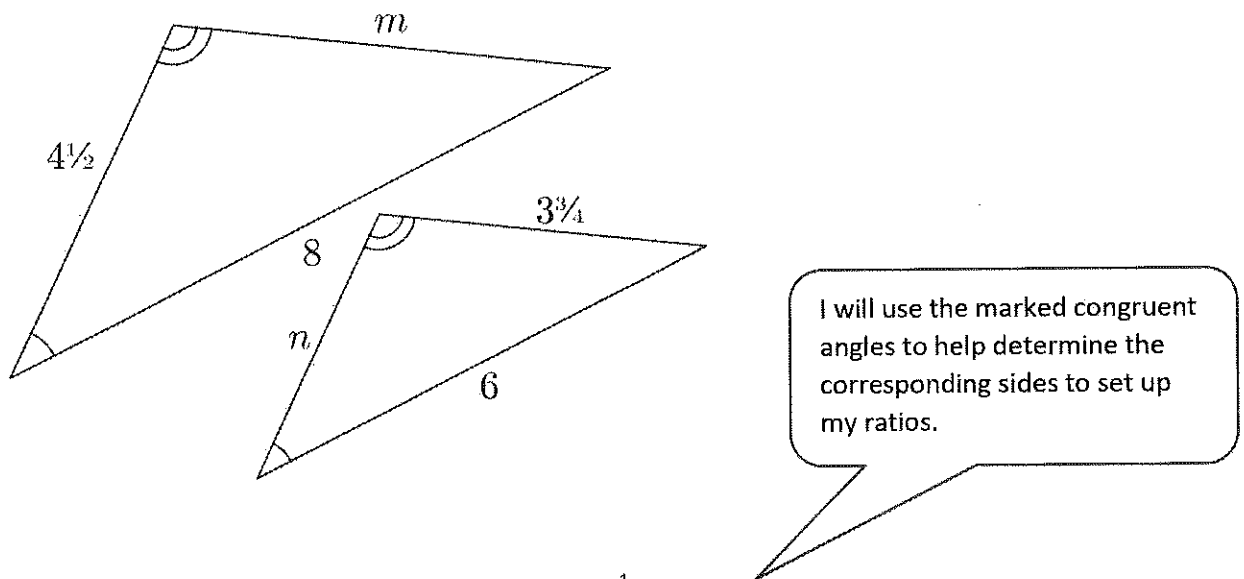
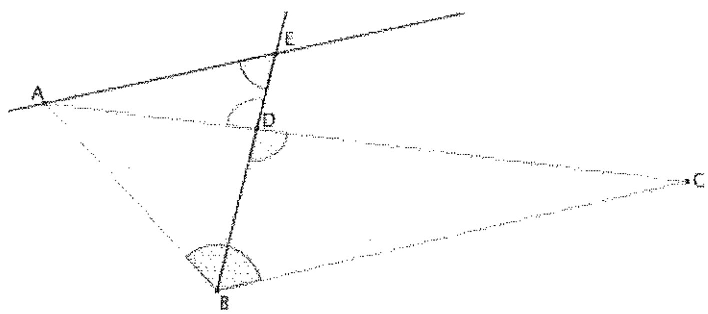
4. For the pair of similar triangles below, determine the unknown lengths of the sides labeled with letters.



The ratios of corresponding sides must be equal, so $\frac{4\frac{1}{2}}{n} = \frac{8}{6}$, ***giving*** $n = 3\frac{3}{8}$. ***Likewise,*** $\frac{m}{3\frac{3}{4}} = \frac{8}{6}$, ***giving*** $m = 5$.

Lesson 18: Similarity and the Angle Bisector Theorem

1. In the following diagram, $\overrightarrow{BD}$ bisects $\angle B$, and $\overleftrightarrow{AE}$ is parallel to $\overleftrightarrow{BC}$. By the angle bisector theorem, since $\overrightarrow{BD}$ bisects $\angle B$, the value of the ratio $AB: CB$ should be equal to the value of the ratio $AD: CD$. Answer the following guided questions that help illustrate why this is true.



a. We can get close to showing that $AB: CB = AD: CD$ by showing that $CB: CD = AE: AD$, a relationship that can be established once it is shown that $\triangle BDC \sim \triangle EDA$. What information establishes that $\triangle BDC \sim \triangle EDA$?

$\angle BDC$ and $\angle EDA$ are equal in measure since they are vertical angles, and $\angle DBC$ and $\angle DEA$ are equal in measure since they are alternate interior angles along parallel lines cut by a transversal. Then by the AA theorem, $\triangle BDC \sim \triangle EDA$. Since the triangles are similar, their corresponding sides have proportional lengths.

It is helpful to clearly mark any angles of equal measure in such a diagram.

b. Why is it important to establish the relationship $CB: CD = AE: AD$?

Out of the four lengths in the relationship, three lengths (CB, CD, and AD) are in the relationship we are trying to demonstrate ($AB: CB = AD: CD$).

c. How can we establish a relationship that yields information on the length of $\overline{AB}$?

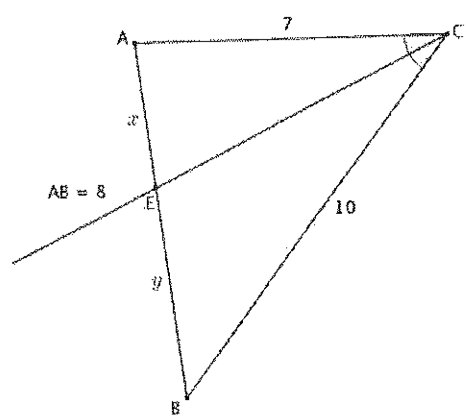
We know that $\angle DBC$ and $\angle DEA$ are equal in measure (alternate interior angles) and that the measures of $\angle ABD$ and $\angle DBC$ are equal since they are the bisected halves of $\angle B$. This means that $\angle ABD$ and $\angle DEA$ are equal in measure. By the converse of the base angles of an isosceles triangle theorem, it follows that $\triangle ABE$ is isosceles, and $AB = AE$.

d. Is it now possible to show that $AB: CB = AD: CD$?

Since $AB = AE$, AB can be substituted into the relationship in part (b) so that $CB: CD = (AB): AD$. This can be rewritten as $AB: CB = AD: CD$.

Rewriting this relationship will require several algebraic properties.

2. In the following triangle, $\overrightarrow{CE}$ bisects $\angle C$ and intersects $\overline{AB}$ at point E. Find the lengths of x and y.



$$x + y = 8$$
$$x = 8 - y$$

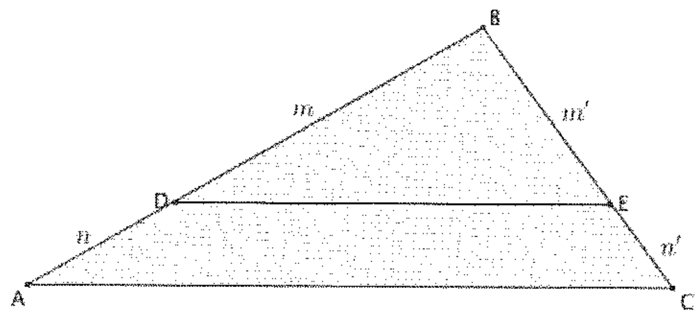
$$\frac{7}{10} = \frac{x}{y}$$
$$7y = 10x$$
$$7y = 10(8 - y)$$
$$7y = 80 - 10y$$
$$17y = 80$$
$$y = \frac{80}{17} = 4\frac{12}{17}$$

$$x = 8 - \left(4\frac{12}{17}\right) = 3\frac{5}{17}$$

The length of x is $3\frac{5}{17}$, and the length of y is $4\frac{12}{17}$.

Lesson 19: Families of Parallel Lines and the Circumference of the Earth

1. In the following triangle, $\overline{DE} \parallel \overline{AC}$.



   a. Does $\overline{DE}$ split the sides $\overline{AB}$ and $\overline{BC}$ proportionally?

   By the triangle side splitter theorem, since $\overline{DE}$ is parallel to $\overline{AC}$, it also splits sides $\overline{AB}$ and $\overline{BC}$ proportionally.

   b. Show that $m: m' = n: n'$ is equivalent to $m: n = m': n'$.

$$\frac{m}{m'} = \frac{n}{n'}$$
$$mn' = m'n$$
$$\frac{mn'}{nn'} = \frac{m'n}{nn'}$$
$$\frac{m}{n} = \frac{m'}{n'}$$

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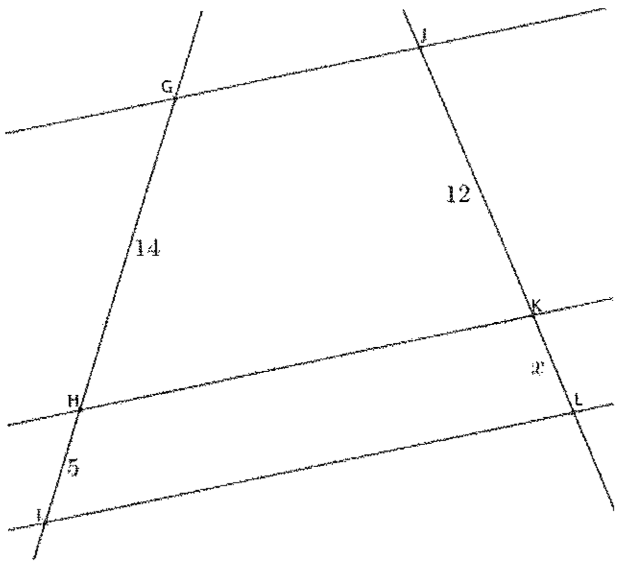
2. In the following figure, lines that appear to be parallel are in fact parallel. Show two different solutions to solve for the length x.



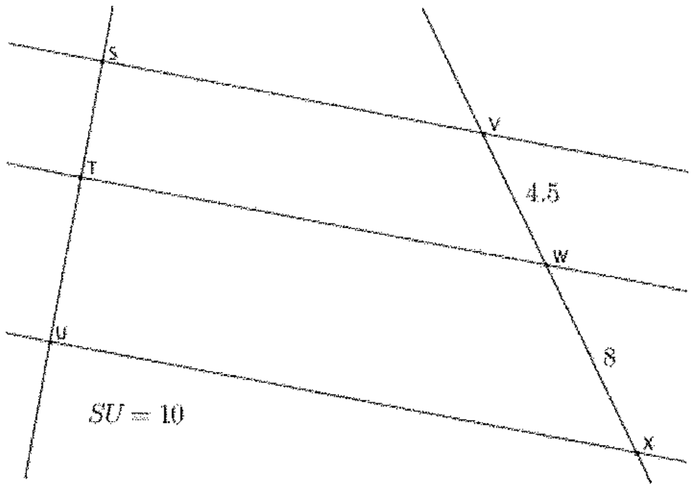
> I must remember the following theorem: If parallel lines are intersected by two transversals, then the ratios of the segments determined along each transversal between the parallel lines are equal.

$$\frac{14}{5} = \frac{12}{x}$$
$$14x = 60$$
$$x = \frac{30}{7}$$

$$\frac{14}{12} = \frac{5}{x}$$
$$14x = 60$$
$$x = \frac{30}{7}$$

The length x is equal to $\frac{30}{7}$.

3. In the following figure, lines that appear to be parallel are in fact parallel. $VW = 4.5$, $WX = 8$, and $SU = 10$. Find ST and TU.



Let TU be x, and ST be $(10 - x)$.

$$\frac{4.5}{8} = \frac{10 - x}{x}$$
$$80 - 8x = 4.5x$$
$$12.5x = 80$$
$$x = 6.4$$

TU is equal to 6.4, and ST is equal to $(10 - (6.4))$, or 3.6.

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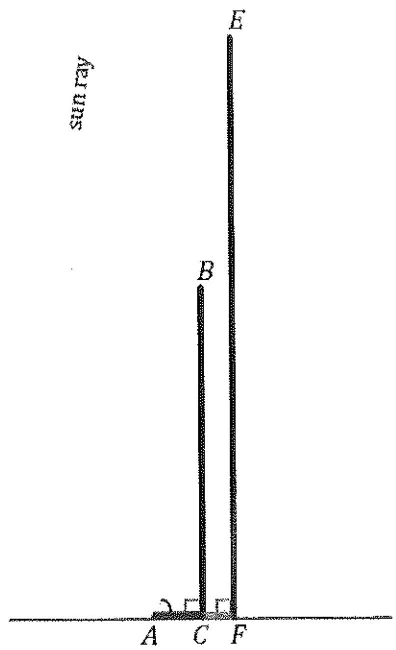
4. In Eratosthenes' quest to determine the circumference of the earth, we hypothesized that he found the angle between the sun's rays and the perpendicular to the ground by using a pole (assumed to also be perpendicular to the ground). Using the pole's shadow and a tool like a protractor, he measured the angle to be 7.2°. Why is the height of the pole in this measurement irrelevant? Use a diagram to support your answer.



The height of the pole used in the measurement is irrelevant because the combination of the way the sun's rays meet the ground and the perpendicularity of a pole to the ground together guarantee similar triangles formed by the rays, poles, and shadows of poles. As shown in the diagram, $\triangle ABC$ and $\triangle AEF$ each share an angle at $\angle A$, and each triangle has a $90°$ angle. Then, $\triangle ABC$ and $\triangle AEF$ are similar by the AA theorem. The respective shadows and poles are proportional in length. Therefore, regardless of the length of the pole, the use of its shadow will help determine the $7.2°$ angle formed by the sun's ray and the pole.

Lesson 20: How Far Away Is the Moon?

1. In order for a spherical object to appear as though it is *just* blocking out the sun, the object must be held from the eye at a length of 108 times the diameter of the spherical object. Approximately how many feet from the eye would the following objects need to be to just block out the sun:

   a. A basketball?

   A basketball has a diameter of approximately 9.55 *inches. To just block out the sun, it must be a distance of* $108(9.55)$ *inches from the eye, or approximately* 86 *feet from the eye.*

   b. A ping pong ball?

   A ping pong ball has a diameter of approximately 1.57 *inches. To just block out the sun, it must be a distance of* $108(1.57)$ *inches from the eye, or approximately* 14 *feet from the eye.*

   c. Describe what would happen if you held the basketball a distance less than the answer you found in part (a). Describe what would happen if held it a distance greater than what you found in part (a).

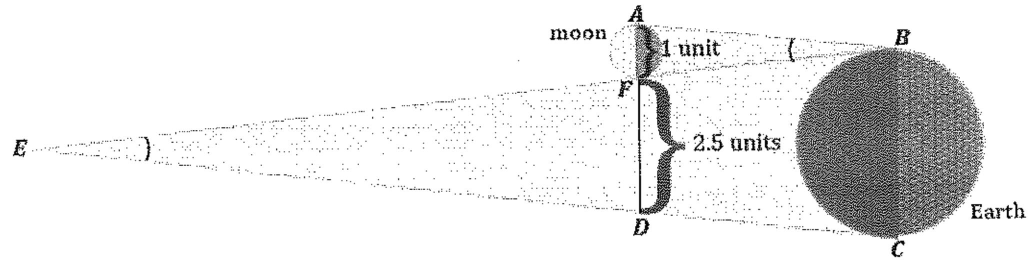
   If the basketball were held less than 86 *feet from the eye, it would more than block out the sun, while holding it at a distance greater than* 86 *feet would not block out the sun. Instead, the basketball would appear as a dark disk against the light of the sun. Do not attempt this!*

   d. Since the moon just blocks out the sun during a solar eclipse, what can be concluded about the distance of the moon from Earth?

   The distance of the moon from Earth must be 108 *times the moon's diameter.*

> During a solar eclipse, the moon passes between the earth and the sun. During a lunar eclipse, the earth passes between the sun and the moon.

2. Based on a few assumptions made and a clever model, the ancient Greeks were able to approximate the distance between the moon and the earth. Use what you learned in class to answer the following questions on the model the Greeks developed.



a. What does the term "1 unit" refer to?

The diameter of the moon

b. How was it established that $DF = 2.5$ units?

Observation of the time needed for the moon to pass through the earth's shadow during a total lunar eclipse yielded that the diameter of the cross section of the earth's conical shadow at the distance of the moon is about $2\frac{1}{2}$ moon diameters.

c. It is critical to establish $ABCD$ as a parallelogram in order to approximate the distance between the moon and the earth. Explain how the quadrilateral can be determined as a parallelogram.

It can be shown that the opposite sides of the quadrilateral are both parallel.

$\overline{AD} \parallel \overline{BC}$ since the diameter of the earth is parallel to the segment formed by the moon's diameter and the diameter of the shadow at the distance of the moon.

It is assumed that the conical shadows of celestial bodies are similar; therefore, the two-dimensional profile view, which are isosceles triangles, are also similar. Then, $m\angle ABF = m\angle BEC$ since the shadows are similar triangles, which implies that $\overline{AB} \parallel \overline{CD}$ (alternate interior angles).

d. How was the earth's diameter (roughly 8,000 miles) used to approximate the distance to the moon?

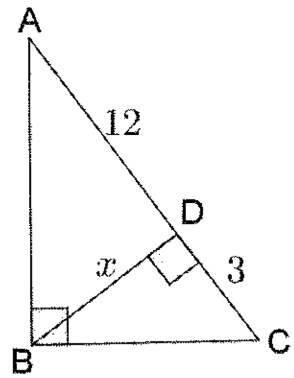
Once it was established that $ABCD$ was a parallelogram, and $AD = 3.5$ units, the following could be written:

$$3.5 \text{ units} \approx 8000 \text{ miles}$$
$$1 \text{ unit} \approx 2300 \text{ miles}$$

This means that the diameter of the moon was approximated at $2,300$ miles, and the distance to the earth was 108 times that, or $108(2300)$ miles, or approximately $248,000$ miles.

Lesson 21: Special Relationships Within Right Triangles—Dividing into Two Similar Sub-Triangles

1. Use similar triangles to find the length of the altitudes labeled with variables in each triangle below.



$\triangle ADB \sim \triangle BDC$ by AA criterion, so corresponding sides are proportional.

$$\frac{x}{3} = \frac{12}{x}$$
$$x^2 = 36$$
$$x = \sqrt{36}$$
$$x = 6$$

I can use the cutouts we made in class to help determine the correct ratios.

2. Given right triangle EHG with altitude $\overline{FH}$ drawn to the hypotenuse, find the lengths of $\overline{FH}$, $\overline{FE}$, and $\overline{FG}$. Note: $EG = 18$.

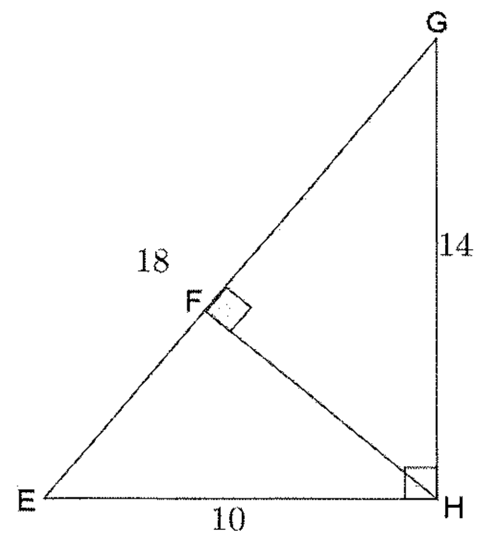
The altitude drawn from F to H cuts triangle EGH into two similar sub-triangles providing the following correspondence:

$$\triangle EHG \sim \triangle EFH \sim \triangle HFG$$

When writing a similarity statement, I like to keep the right angle as the middle letter.

Using the shorter leg:hypotenuse ratio for the similar triangles,

$$\frac{10}{18} = \frac{FH}{14} \qquad \frac{10}{18} = \frac{FE}{10}$$
$$140 = 18(FH) \qquad 100 = 18(FE)$$
$$\frac{140}{18} = FH \qquad \frac{100}{18} = FE$$
$$7\frac{14}{18} = FH \qquad 5\frac{10}{18} = FE$$
$$7\frac{7}{9} = FH \qquad 5\frac{5}{9} = FE$$



By addition:

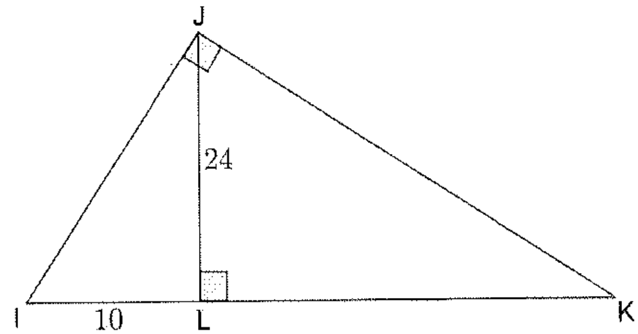
$$FE + FG = EG$$
$$5\frac{5}{9} + FG = 18$$
$$FG = 12\frac{4}{9}$$

3. Given triangle IJK with altitude $\overline{JL}$, $JL = 24$, and $IL = 10$, find IJ, JK, LK, and IK.

Altitude $\overline{JL}$ cuts $\triangle IJK$ into two similar sub-triangles such that $\triangle IJK \sim \triangle ILJ \sim \triangle JLK$.

By the Pythagorean theorem:

$$10^2 + 24^2 = IJ^2$$
$$100 + 576 = IJ^2$$
$$676 = IJ^2$$
$$\sqrt{676} = IJ$$
$$26 = IJ$$



Using the shorter leg:longer leg ratio,

$$\frac{10}{24} = \frac{26}{JK}$$
$$10(JK) = 624$$
$$JK = \frac{624}{10}$$
$$JK = 62\frac{2}{5}$$

Using the shorter leg:hypotenuse ratio,

$$\frac{10}{26} = \frac{26}{IK}$$
$$10(IK) = 676$$
$$IK = \frac{676}{10}$$
$$IK = 67\frac{3}{5}$$

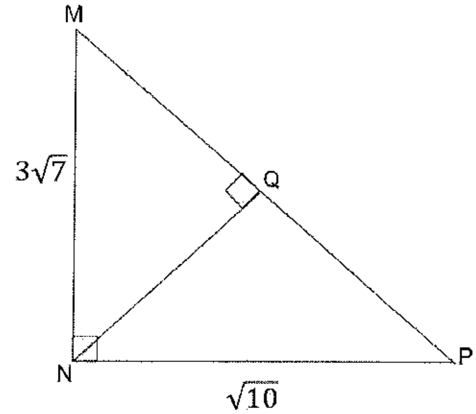
Using addition,

$$IL + LK = IK$$
$$10 + LK = 67\frac{3}{5}$$
$$LK = 57\frac{3}{5}$$

Since they are all similar right triangles, I can use the Pythagorean theorem to double check my answers.

4. Given right triangle MNP with altitude $\overline{NQ}$, find NQ, MQ, PQ, and MP.

Using the Pythagorean theorem,



$$(3\sqrt{7})^2 + (\sqrt{10})^2 = MP^2$$
$$63 + 10 = MP^2$$
$$73 = MP^2$$
$$\sqrt{73} = MP$$

Since 73 is not a perfect square and has no perfect square factors, the answer is $\sqrt{73}$.

An altitude from the right angle in a right triangle to the hypotenuse cuts the triangle into two similar right triangles, $\triangle MNP \sim \triangle MQN \sim \triangle NQP$.

Using the shorter leg:hypotenuse ratio,

$$\frac{\sqrt{10}}{\sqrt{73}} = \frac{NQ}{3\sqrt{7}}$$
$$3\sqrt{70} = NQ(\sqrt{73})$$
$$\frac{3\sqrt{70}}{\sqrt{73}} = NQ$$

$$\frac{\sqrt{10}}{\sqrt{73}} = \frac{QP}{\sqrt{10}}$$
$$10 = QP(\sqrt{73})$$
$$\frac{10}{\sqrt{73}} = QP$$

Using addition,

$$MQ + QP = MP$$
$$MQ + \frac{10}{\sqrt{73}} = \sqrt{73}$$
$$MQ = \sqrt{73} - \frac{10}{\sqrt{73}}$$
$$MQ = \frac{63}{\sqrt{73}}$$

Lesson 22: Multiplying and Dividing Expressions with Radicals

Express each number in its simplest radical form.

Using Rule 1, I can rewrite $\sqrt{15}$ as $\sqrt{3}\cdot\sqrt{5}$ so that I can simplify.

1. $\sqrt{3}\cdot\sqrt{15}=$

$$\sqrt{3}\cdot\sqrt{15}=\sqrt{3}\cdot\sqrt{3}\cdot\sqrt{5}$$
$$=3\sqrt{5}$$

2. $\sqrt{450}=$

I want to rewrite the number under the radical sign, 450, as products of perfect squares, if possible.

$$\sqrt{450}=\sqrt{25}\cdot\sqrt{9}\cdot\sqrt{2}$$
$$=5\cdot 3\sqrt{2}$$
$$=15\sqrt{2}$$

I could rewrite $\sqrt{x^4}$ as $\sqrt{(x^2)^2}$.

3. $\sqrt{24x^5}=$

$$\sqrt{24x^5}=\sqrt{4}\sqrt{6}\sqrt{x^4}\sqrt{x}$$
$$=2x^2\sqrt{6x}$$

4. $\sqrt{105}=$

The number 105 can be factored, but none of the factors are perfect squares, which are necessary to simplify. Therefore, $\sqrt{105}$ cannot be simplified.

5. Show and explain that $\frac{2}{\sqrt{3}}$ and $\frac{2\sqrt{3}}{3}$ are equivalent.

I need to rationalize the denominator to write the second expression in simplest radical form.

$$\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$
$$=\frac{2\sqrt{3}}{3}$$

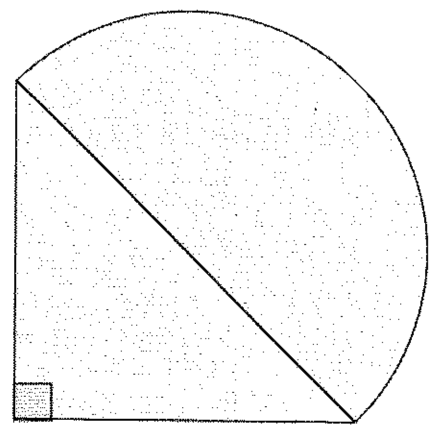
By rationalizing the denominator, I can show the two expressions are equivalent.

6. $\sqrt{\frac{3}{32}} =$

$$\sqrt{\frac{3}{32}} = \frac{\sqrt{3}}{\sqrt{32}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{6}}{\sqrt{64}}$$
$$= \frac{\sqrt{6}}{8}$$

I could multiply $\frac{\sqrt{3}}{\sqrt{32}}$ by $\frac{\sqrt{32}}{\sqrt{32}}$ or $\frac{\sqrt{8}}{\sqrt{8}}$ or $\frac{\sqrt{2}}{\sqrt{2}}$ to get the result of a perfect square under the radical sign in the denominator.

7. Determine the exact area of the shaded region that consists of a triangle and a semicircle, shown to the right.



The area of the triangle is

$A = \frac{1}{2}(\sqrt{5})(\sqrt{5}) = \frac{1}{2}(5) = \frac{5}{2}.$

Let d be the length of the hypotenuse.

By special triangles or the Pythagorean theorem, $d = \sqrt{10}$.

The radius of the semicircle is $\frac{1}{2}\sqrt{10} = \frac{\sqrt{10}}{2}$.

The area of the semicircle is

$$A = \frac{1}{2}\pi\left(\frac{1}{2}\sqrt{10}\right)^2$$
$$= \frac{1}{2}\pi\left(\frac{1}{4}\right)(10)$$
$$= \frac{5\pi}{4}.$$

$$\text{Total(Area)} = \frac{5}{2} + \frac{5\pi}{4}$$
$$= \frac{10}{4} + \frac{5\pi}{4}$$
$$= \frac{10 + 5\pi}{4}$$

The sum of the areas of the triangle and the semicircle is $\frac{10+5\pi}{4}$ square units.

Lesson 23: Adding and Subtracting Expressions with Radicals

Express each answer in simplified radical form.

1. $22\sqrt{3} - 18\sqrt{3} =$

$$\begin{aligned} 22\sqrt{3} - 18\sqrt{3} &= (22 - 18)\sqrt{3} \\ &= 4\sqrt{3} \end{aligned}$$

I will use the distributive property to add or subtract expressions with radicals.

2. $\sqrt{18} + 5\sqrt{2} =$

I need to rewrite $\sqrt{18}$ so I can apply the distributive property.

$$\begin{aligned} \sqrt{18} + 5\sqrt{2} &= 3\sqrt{2} + 5\sqrt{2} \\ &= (3 + 5)\sqrt{2} \\ &= 8\sqrt{2} \end{aligned}$$

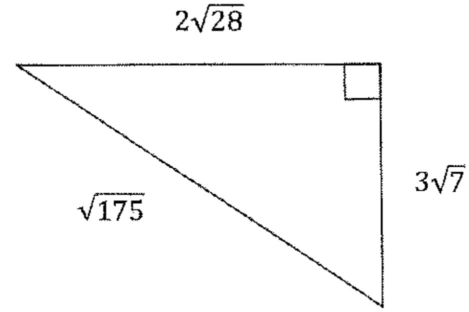
3. $4\sqrt{5} + 11\sqrt{125} =$

$$\begin{aligned} 4\sqrt{5} + 11\sqrt{125} &= 4\sqrt{5} + 11\sqrt{25}\sqrt{5} \\ &= 4\sqrt{5} + 11(5)\sqrt{5} \\ &= 4\sqrt{5} + 55\sqrt{5} \\ &= (4 + 55)\sqrt{5} \\ &= 59\sqrt{5} \end{aligned}$$

4. Determine the area and perimeter of the triangle shown. Simplify as much as possible.

Perimeter:

$$\begin{aligned}\sqrt{175}+2\sqrt{28}+3\sqrt{7} &= \sqrt{25}\sqrt{7}+2\sqrt{4}\sqrt{7}+3\sqrt{7}\\ &= 5\sqrt{7}+2\cdot 2\sqrt{7}+3\sqrt{7}\\ &= 5\sqrt{7}+4\sqrt{7}+3\sqrt{7}\\ &= (5+4+3)\sqrt{7}\\ &= 12\sqrt{7}\end{aligned}$$



Area:

$$\frac{2\sqrt{28}(3\sqrt{7})}{2}=\frac{4\sqrt{7}(3\sqrt{7})}{2}=\frac{84}{2}=42$$

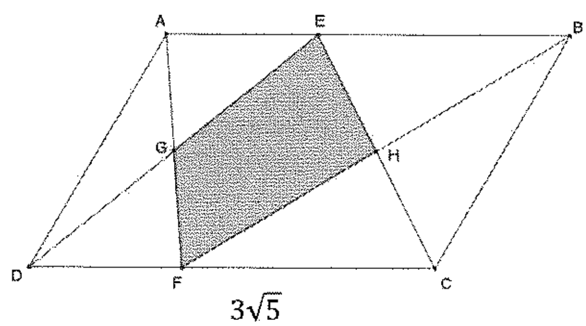
The perimeter is $12\sqrt{7}$ units, and the area is 42 square units.

5. Parallelogram $ABCD$ has an area of $12\sqrt{5}$ square units. $DC = 3\sqrt{5}$, and G and H are midpoints of $\overline{DE}$ and $\overline{CE}$, respectively. Find the area of the shaded region. Write your answer in simplest radical form.

Using the area of a parallelogram,

$$\begin{aligned}\text{Area}(ABCD) &= bh\\ 12\sqrt{5} &= 3\sqrt{5}\cdot h\\ 4 &= h\end{aligned}$$

The height of the parallelogram is 4 units.



The area of the shaded region is the sum of the areas of $\triangle EGH$ and $\triangle FGH$.

The given points G and H are midpoints of $\overline{DE}$ and $\overline{CE}$; therefore, by the triangle side splitter theorem, $\overline{GH}$ must be parallel to $\overline{DC}$, and thus, also parallel to $\overline{AB}$. Furthermore,
$GH = \frac{1}{2}DC = \frac{1}{2}AB = \frac{3}{2}\sqrt{5}$.

I learned about the side splitter theorem in Lesson 4 and have been using it ever since.

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$\triangle EGH \sim \triangle EDC$ by AA criterion with a scale factor of $\frac{1}{2}$. The areas of scale drawings are related by the square of the scale factor; therefore, $\text{Area}(\triangle EGH) = \left(\frac{1}{2}\right)^2 \cdot \text{Area}(\triangle EDC)$.

$$\text{Area}(\triangle EDC) = \frac{1}{2} \cdot 3\sqrt{5} \cdot 4$$
$$\text{Area}(\triangle EDC) = 6\sqrt{5}$$

My plan is to determine $\text{Area}(EHFG) = \text{Area}(\triangle EGH) + \text{Area}(\triangle FGH)$.

$$\text{Area}(\triangle EGH) = \left(\frac{1}{2}\right)^2 \cdot 6\sqrt{5}$$
$$\text{Area}(\triangle EGH) = \frac{1}{4} \cdot 6\sqrt{5}$$
$$\text{Area}(\triangle EGH) = \frac{3}{2}\sqrt{5}$$

By a similar argument,

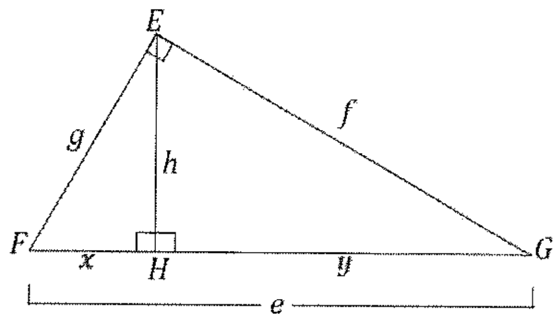
$$\text{Area}(\triangle FGH) = \frac{3}{2}\sqrt{5}$$

$$\text{Area}(EHFG) = \text{Area}(\triangle EGH) + \text{Area}(\triangle FGH)$$
$$\text{Area}(EHFG) = \frac{3}{2}\sqrt{5} + \frac{3}{2}\sqrt{5}$$
$$\text{Area}(EHFG) = 3\sqrt{5}$$

The area of the shaded region is $3\sqrt{5}$ *square units.*

Lesson 24: Prove the Pythagorean Theorem Using Similarity

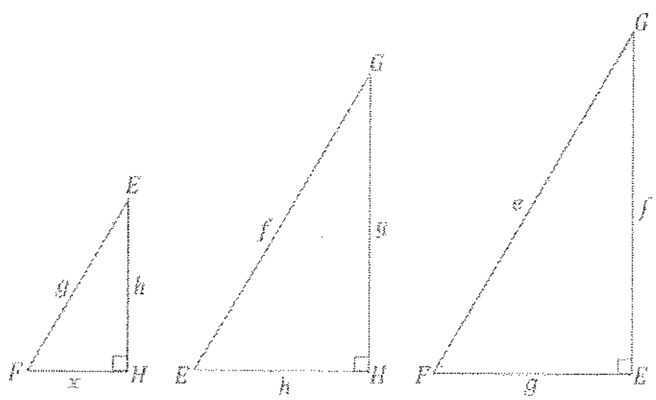
1. In right triangle EFG, an altitude has been drawn from vertex E and meets $\overline{FG}$ at H. Answer the following questions based on this figure. H divides $\overline{FG}$ into the lengths x and y.



a. $\triangle EFG \sim \triangle HFE \sim \triangle HEG$. Explain why.

$\triangle EFG \sim \triangle HFE$ by the AA criterion, as both triangles have a $90°$ angle, and both triangles share $\angle F$. $\triangle EFG \sim \triangle HEG$, also by the AA criterion, as both triangles have a $90°$ angle, and both triangles share $\angle G$. Similarity is transitive; therefore, $\triangle HFE \sim \triangle HEG$.

b. Sketch each of the three triangles separately so that their corresponding sides and angles are easy to see.



To help guide your sketches, consider beginning with the right angle for each triangle to help orient you in the sketching process.

In order to prove the Pythagorean theorem, we want to show that the sum of each leg length, squared, is equal to the hypotenuse, squared, or based on this figure, $f^2 + g^2 = e^2$. To prove the Pythagorean theorem using similarity, we will employ the fact that the ratios of corresponding proportional lengths are equal in value between similar triangles.

c. $\triangle EFG$ and $\triangle HFE$ each have $\angle F$ as an acute angle. Write the equivalent shorter leg:hypotenuse ratios for these triangles.

$$\frac{x}{g} = \frac{g}{e}$$
$$g^2 = xe$$

d. $\triangle EFG$ and $\triangle HEG$ each have $\angle G$ as an acute angle. Write the equivalent longer leg:hypotenuse ratios for these triangles.

$$\frac{y}{f} = \frac{f}{e}$$
$$f^2 = ye$$

e. Use parts (c) and (d) to show that $f^2 + g^2 = e^2$.

$$\begin{aligned} f^2 + g^2 &= xe + ye \\ &= e(x+y) \\ &= e(e) \\ &= e^2 \end{aligned}$$

2. What is the leg:hypotenuse ratio of a 45–45–90 triangle? How do you know?

Take for example a 45–45–90 triangle with leg lengths of 1 each. Let the hypotenuse be a length c. Then by the Pythagorean theorem:

$$1^2 + 1^2 = c^2$$
$$2 = c^2$$
$$c = \sqrt{2}$$

The leg:hypotenuse ratio is $1:\sqrt{2}$. By the AA criterion, triangles with angle measures 45–45–90 are similar and, therefore, corresponding lengths are proportional. This means that all 45–45–90 triangles will have a leg:hypotenuse ratio of $1:\sqrt{2}$.

3. Consider any 30–60–90 triangle.

a. Write the three ratios that compare the lengths of each pair of sides in a 30–60–90 triangle.

The shorter leg:hypotenuse ratio is $1:2$. The longer leg:hypotenuse ratio is $\sqrt{3}:2$. The shorter leg:hypotenuse ratio is $1:\sqrt{3}$.

b. By the AA criterion, all triangles with angle measures of $30–60–90$ are similar. Use the ratios from part (a) to find the leg lengths of a $30–60–90$ triangle with a hypotenuse of 100.

Let a represent the length of the shorter leg. Since the shorter leg:hypotenuse ratio is $1:2$, then

$$\frac{a}{100} = \frac{1}{2}$$
$$a = 50.$$

Let b represent the length of the longer leg. Since the longer leg:hypotenuse ratio is $\sqrt{3}:2$, then

$$\frac{b}{100} = \frac{\sqrt{3}}{2}$$
$$2b = 100\sqrt{3}$$
$$b = 50\sqrt{3}.$$

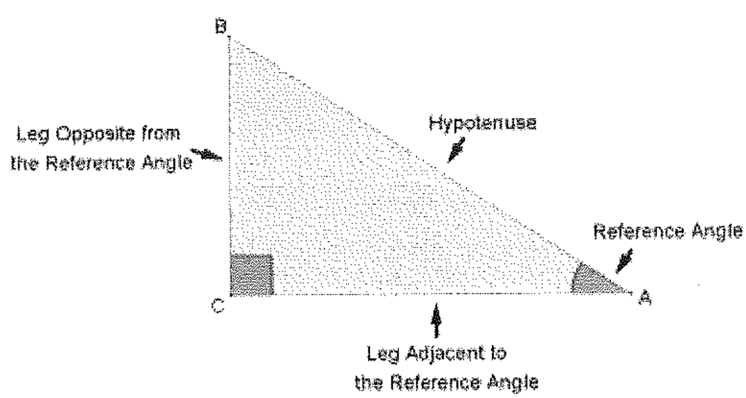
Lesson 25: Incredibly Useful Ratios

Problem Set

The table below contains the values of the ratios $\frac{opp}{hyp}$ and $\frac{adj}{hyp}$ for a variety of right triangles based on a given acute angle, θ, from each triangle. Use the table and the diagram of the right triangle below to complete each problem.

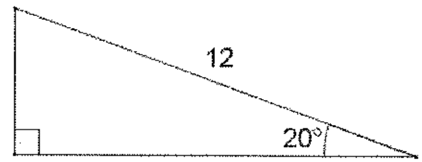
The abbreviation adj refers to the *adjacent leg*, or the length of the leg that lies *adjacent* to the acute angle being referenced. The abbreviation opp refers to the *opposite leg*, or the length of the leg that lies *opposite* the acute angle being referenced. Finally, the abbreviation hyp refers to the length of the *hypotenuse* of the right triangle.

θ (degrees)	0	10	20	30	40	45	50	60	70	80	90
$\frac{opp}{hyp}$	0	0.1736	0.3420	$\frac{1}{2}=0.5$	0.6428	0.7071	0.7660	0.8660	0.9397	0.9848	1
$\frac{adj}{hyp}$	1	0.9848	0.9397	0.8660	0.7660	0.7071	0.6428	$\frac{1}{2}=0.5$	0.3420	0.1736	0



For each problem, approximate the unknown lengths to one decimal place. Write the values of the ratios of appropriate lengths using the terms opp, adj, and hyp.

1. Find the approximate length of the leg opposite the $20°$ angle.



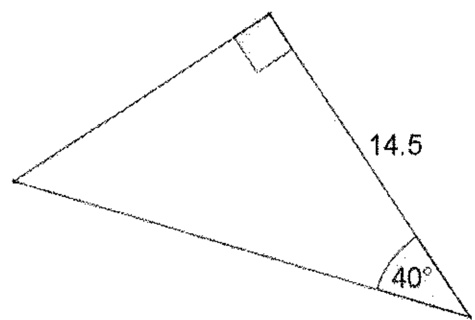
From the table, the ratio value $\frac{opp}{hyp}$ *for a* $20°$ *angle is* 0.3420.

$$\frac{opp}{12} = 0.3420$$
$$12 \cdot \left(\frac{opp}{12}\right) = 12 \cdot (0.3420)$$
$$opp = 4.104$$
$$opp \approx 4.1$$

Rounded to one decimal place, the length of the leg that lies opposite the $20°$ *angle is approximately* 4.1.

The side that has a length of 12 is across from the right angle, so it must be the hypotenuse. If I need to find the unknown length of the *opposite leg,* then I know that the ratio to use is $\frac{opp}{hyp}$ for a $20°$ angle.

2. Find the approximate length of the hypotenuse.



From the table, the ratio value $\frac{adj}{hyp}$ *for a* $40°$ *angle is* 0.7660.

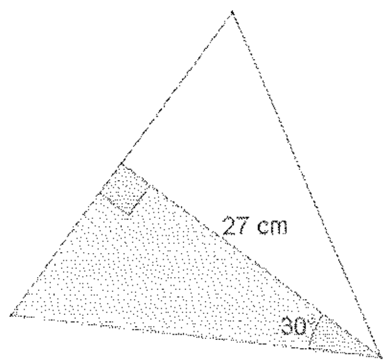
$$\frac{14.5}{hyp} = 0.7660$$
$$hyp = \frac{14.5}{0.7660}$$
$$hyp \approx 18.9$$

Rounded to one decimal place, the length of the hypotenuse is approximately 18.9.

I am given an acute angle of $40°$ and a leg of length 14.5. The leg with the known length makes up one side of the given angle, so the leg is *adjacent* to the $40°$ angle. To solve this problem, I will use the ratio value $\frac{adj}{hyp}$ for a $40°$ angle.

3. An altitude drawn in an equilateral triangle bisects one of its angles. If the altitude is 27 cm in length, what is the perimeter of the equilateral triangle to the nearest tenth of a centimeter?

An equilateral triangle contains three angles each measuring $60°$. An altitude drawn from one vertex cuts the angle into two adjacent $30°$ angles and meets the opposite side at a $90°$ angle.



An altitude of a triangle forms a right angle with one side of the equilateral triangle giving me a right triangle. I know that the angles in an equilateral triangle are all 60°, so the angle that is bisected must be 30°. Aha! I can use the $\frac{adj}{hyp}$ ratio value for a 30° angle from the table to find the length of the sides of the equilateral triangle.

From the table, the ratio value $\frac{adj}{hyp}$ for a $30°$ angle is 0.8660.

$$\frac{27}{hyp} = 0.8660$$

$$hyp \cdot \left(\frac{27}{hyp}\right) = hyp \cdot (0.8660)$$

$$27 = hyp \cdot (0.8660)$$

$$\frac{1}{0.8660} \cdot (27) = \frac{1}{0.8660} \cdot (hyp \cdot 0.8660)$$

$$\frac{27}{0.8660} = hyp$$

This quotient has a long decimal expansion, and I do not want to approximate until the end of the problem, so I will leave this in quotient form for now.

The length of each side of the equilateral triangle is $\frac{27}{0.8660}$ cm. The perimeter of the equilateral triangle is then three times that length.

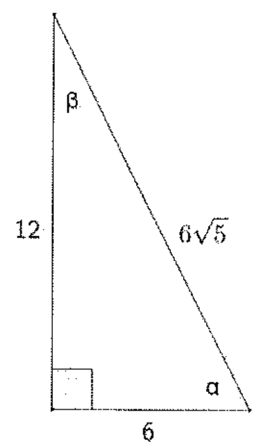
$$\text{Perimeter} = 3 \cdot \left(\frac{27}{0.8660}\right)$$

$$\text{Perimeter} = \frac{81}{0.8660} \approx 93.5$$

The perimeter of the equilateral triangle is approximately 93.5 cm.

Lesson 26: The Definition of Sine, Cosine, and Tangent

1. Given the triangle in the diagram, complete the following table. Simplest radical form is not required.



> The ratio values are dependent upon which acute angle in the triangle is being referenced. To determine the correct values, I will consider the *opp*, *adj*, and *hyp* with respect to the angle marked α first to get the sin, cos, and tan values. Next I will reconsider the *opp*, *adj*, and *hyp* with respect to the angle marked β because the side that lies opposite to the angle α lies adjacent to the angle β.

Angle Measure	Sin	Cos	Tan
α	$\frac{12}{6\sqrt{5}}$	$\frac{6}{6\sqrt{5}}$	$\frac{12}{6} = 2$
β	$\frac{6}{6\sqrt{5}}$	$\frac{12}{6\sqrt{5}}$	$\frac{6}{12} = \frac{1}{2}$

> Remember that: $\tan\theta = \frac{opp}{adj}$

2. Complete the missing values in the table using the relationships of sine, cosine, and tangent.

Angle Measure	Sin	Cos	Tan
α	$\frac{2}{\sqrt{29}}$	$\frac{5}{\sqrt{29}}$	$\frac{2}{5}$
β	$\frac{5}{\sqrt{29}}$	$\frac{2}{\sqrt{29}}$	$\frac{5}{2}$

> α and β are the two acute angles in a right triangle, so they are complements of each other. I can see in the table above that $\sin\alpha = \cos\beta$, $\cos\alpha = \sin\beta$, and $\tan\alpha$ and $\tan\beta$ are reciprocals.

3. Travis looks across the Mississippi River and wants to know how far across it is to the other bank at his location. He knows that he could use the Internet to find out, but he wants to see if he can approximate it himself. To begin, he picks a landmark that appears to be in line with his location and perpendicular to the river. Next, he walks parallel to the river 200 walking paces and estimates the distance to be approximately 600 ft from his original location. He looks across the river to the landmark he chose and estimates that the angle formed between the river bank and his line of sight is approximately 85°. Draw a diagram, and use the estimated measurements to approximate the width of the Mississippi River at Travis's location. Do you think Travis's approximation is accurate? Support your answer.

If I consider the banks of the river to be parallel, then the distance across the river is the distance between parallel lines. The distance between parallel lines is measured perpendicularly to the lines. This is where my right angle is formed for a right triangle.

(See the diagram drawn to the right.)

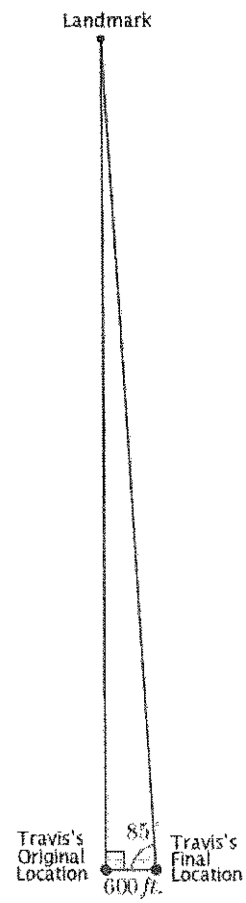
The views that Travis has from his original position and his final position can be represented by a leg and a hypotenuse of a right triangle, respectively. The other leg of the triangle represents the distance that he walked parallel to the river.

The unknown distance is the length of the leg opposite the $85°$ angle.

$$\tan 85 = \frac{opp}{600}$$

$$600 \cdot (\tan 85) = 600 \cdot \left(\frac{opp}{600}\right)$$

$$6858 \approx opp$$



The distance across the Mississippi River, according to Travis's estimated measurements, is approximately 6858 ft. This is approximately 1.3 miles.

According to the National Park Service website (www.nps.gov), the Mississippi River ranges from between 20 and 30 feet wide to over 11 miles wide near Bena, MN. It also states that the widest navigable part of the river is approximately 2 miles in width. Depending on Travis's location along the river, his estimate may or may not be accurate.

Lesson 27: Sine and Cosine of Complementary Angles and Special Angles

1. Find the value of θ that makes each statement true.

   a. $\sin\theta = \cos\theta$

   $\sin\theta = \cos(90-\theta)$, *so by substitution,*

   $$\cos(90-\theta) = \cos\theta$$
   $$90-\theta = \theta$$
   $$90-\theta+\theta = \theta+\theta$$
   $$90 = 2\theta$$
   $$\frac{1}{2}\cdot(90) = \frac{1}{2}\cdot(2\theta)$$
   $$45 = \theta$$

   The value of θ that makes the given equation true is $45°$.

   > To find the unknown value of θ, I write both expressions in the equation in terms of cos (or I could also use sin). I know that $\sin\theta = \cos(90-\theta)$, so I can use substitution to rewrite $\sin\theta$ in terms of cos.

   b. $\cos(\theta+15) = \sin(3\theta)$

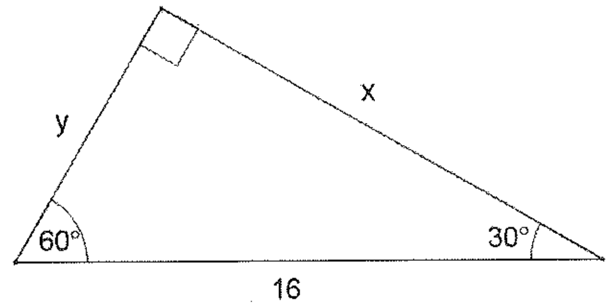
   $\cos\alpha = \sin(90-\alpha)$, so using substitution,

   $$\sin\big(90-(\theta+15)\big) = \sin(3\theta)$$
   $$90-(\theta+15) = 3\theta$$
   $$75-\theta = 3\theta$$
   $$75-\theta+\theta = 3\theta+\theta$$
   $$75 = 4\theta$$
   $$\frac{1}{4}\cdot(75) = \frac{1}{4}\cdot(4\theta)$$
   $$\frac{75}{4} = \theta = 18.75$$

   The value of θ that makes the given equation true is $18.75°$.

   > If I consider the expression $\theta+15$ as a single term such as α, it is easier to see how to substitute the equivalent cos value in its place. Since $\cos\alpha = \sin(90-\alpha)$, when I substitute the value of α back into the equation, I get $\cos(\theta+15) = \sin\big(90-(\theta+15)\big)$.

2. The triangle shown in the diagram to the right is a 30–60–90 triangle. Find the unknown lengths x and y.



$$16 = 2y$$
$$\frac{1}{2} \cdot 16 = \frac{1}{2} \cdot 2y$$
$$8 = y$$

The length of the short leg is 8.

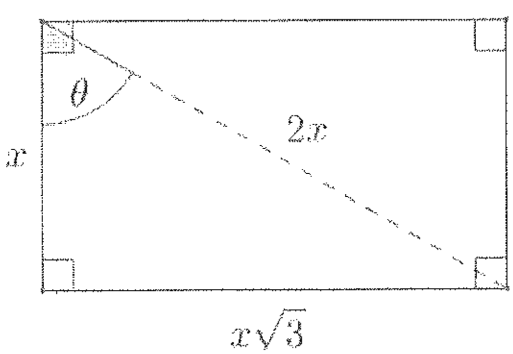
$x = y \cdot \sqrt{3}$

$x = 8 \cdot \sqrt{3}$

The length of the longer leg is $8\sqrt{3}$.

The sides of a 30-60-90 triangle have lengths that are in the ratio $1:\sqrt{3}:2$. The hypotenuse is twice the length of the short leg (leg opposite the 30° angle), and the longer leg (leg opposite the 60° angle) is the length of the shorter leg times $\sqrt{3}$. If I know one of the actual lengths, I can solve for the others by writing equations from the known ratios.

3. A rectangle has a width of x. The length of the rectangle's diagonal is twice its width. Find the area of the rectangle in terms of x.



To find the area of a rectangle, I need its length and width. The only information that I have is that the diagonal is twice the width of the rectangle. The diagonal cuts the rectangle into a pair of right triangles. Aha! If the diagonal is twice the width, this meets the criteria of a 30-60-90 triangle.

The diagonal divides the rectangle into two congruent right triangles. The length of the diagonal is given as twice the width of the rectangle. Since the diagonal is the hypotenuse of a right triangle and the ratio value of $\cos\theta = \frac{x}{2x} = \frac{1}{2}$, the right triangle is a 30–60–90 triangle. Using the ratios of side lengths for such a triangle, the length of the rectangle is $x\sqrt{3}$.

$\text{Area(rectangle)} = \text{length} \cdot \text{width}$

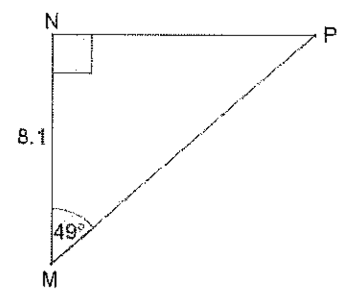
$\text{Area(rectangle)} = x\sqrt{3} \cdot x$

$\text{Area(rectangle)} = x^2\sqrt{3}$

The area of the rectangle is $x^2\sqrt{3}$ square units.

Lesson 28: Solving Problems Using Sine and Cosine

1. Given right triangle MNP with the right angle at N, $MN = 8.1$, and $m\angle M = 49°$, find the measures of the remaining sides and angle to the nearest tenth of a unit.



The two acute angles of a right triangle are complementary.

$m\angle P = 90° - m\angle M$

$m\angle P = 90° - 49°$

$m\angle P = 41°$

The measure of angle P is $41°$.

$\cos 49 = \frac{8.1}{MP}$

$MP = \frac{8.1}{\cos 49}$

$MP \approx 12.3$

The length of hypotenuse $\overline{MP}$ *is approximately* 12.3.

$\tan 49 = \frac{NP}{8.1}$

$NP = 8.1 \cdot \tan 49$

$NP \approx 9.3$

The length of leg $\overline{NP}$ *is approximately* 9.3.

> I can use the interior angle sum of a triangle to find the measure of angle P. Then I can use the value of $\cos 49$ and $\tan 49$ in my calculator and $MN = 8.1$ to find the lengths of $\overline{MP}$ and $\overline{NP}$.

2. In general, is $\sin a + \sin b = \sin(a + b)$ for acute angle measures a and b? Explain why or why not.

In general, the given statement is not true. Consider the following counterexample.

$\sin 20 + \sin 20 = 0.3420201433\ldots + 0.3420201433\ldots$

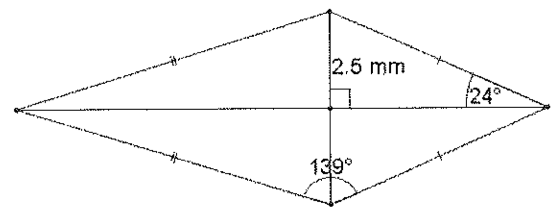
$\sin 20 + \sin 20 = 0.6840402867\ldots$

$\sin 40 = 0.6427876097\ldots$

Therefore, $\sin a + \sin b = \sin(a + b)$ *cannot be considered true in general.*

> To show that a conjecture is false, I only need to find and provide one counterexample. A counterexample is a use of the conjectured relationship that does not hold true.

3. A kite is shown in the diagram to the right. Use your knowledge of angle relationships in geometric figures and trigonometric ratios to find the perimeter of the kite to the nearest tenth of a millimeter.



The leg opposite the given $24°$ *angle is* 2.5 mm *long.*

$$\sin 24 = \frac{2.5}{hyp}$$
$$hyp = \frac{2.5}{\sin 24}$$
$$hyp \approx 6.146$$

Two sides of the kite are approximately 6.146 mm *in length.*

The complement of the $24°$ *angle is* $66°$, *so the* $139°$ *angle is composed of adjacent angles measuring* $66°$ *and* $73°$.

The leg adjacent to the $73°$ *angle is* 2.5 mm.

$$\cos 73 = \frac{2.5}{hyp}$$
$$hyp = \frac{2.5}{\cos 73}$$
$$hyp \approx 8.551$$

The remaining two sides of the kite are approximately 8.551 mm *in length.*

The perimeter of the kite is the sum of all side lengths.

$$\text{Perimeter} = 2 \cdot \left(\frac{2.5}{\sin 24}\right) + 2 \cdot \left(\frac{2.5}{\cos 73}\right)$$
$$\text{Perimeter} = \frac{5}{\sin 24} + \frac{5}{\cos 73} \approx 29.4$$

The perimeter of the kite is approximately 29.4 mm.

> The kite is divided into two pairs of congruent right triangles by its diagonals. I need to find the other angles in the triangles using the angle sum of a triangle and the angle sum of a quadrilateral. Then I can use the trigonometric ratios to find the lengths of the hypotenuses.

> I should use the exact values of the hypotenuses that I found in calculating the perimeter of the kite so that my final answer does not have a significant rounding error.

Lesson 29: Applying Tangents

1. A line on the coordinate plane has an angle of elevation of 75°. Use a slope triangle to show that the slope of the line is equal to the tangent value of the angle of elevation. Then find the slope correct to four decimal places.

Choose two different points on the line such that the horizontal distance between points is $a > 0$. Draw the slope triangle on the chosen points having vertices (x, y), $(x + a, y)$, and $(x + a, y + b)$ for $b > 0$.

$\tan 75 = \frac{(y+b)-y}{(x+a)-x} = \frac{b}{a}$

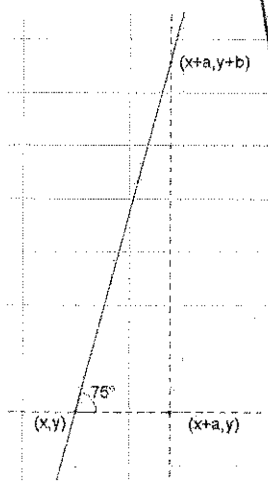
$b = a \tan 75$

$\text{slope} = \frac{\text{rise}}{\text{run}}$

$\text{slope} = \frac{b}{a}$

$\text{slope} = \frac{a \tan 75}{a} = \tan 75$

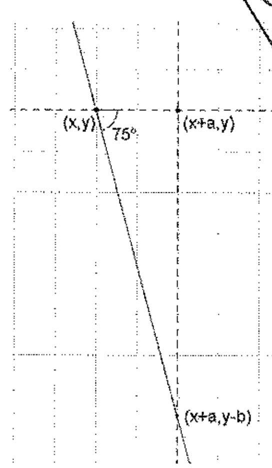
The slope of the line is equal to the tangent of the angle of elevation, $\tan 75$, which is approximately 3.7321.



To find the slope of a line, I need to find the value of the ratio of the rise to run between two different points on the line. A slope triangle along the line will give me legs of a right triangle to work with. If I know the length of one of the legs, then I can use the tangent of 75° to find the other leg length.

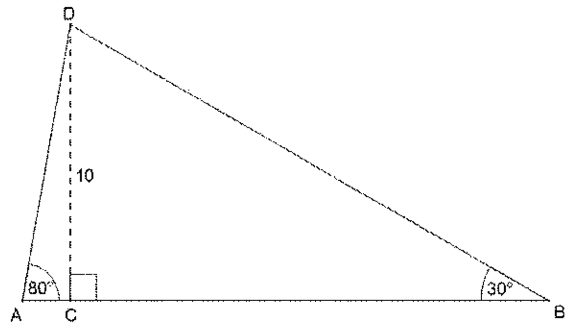
2. Will a line on the coordinate plane with an angle of depression of 75° have a slope equal to the slope of the line in Problem 1? Explain why or why not.

The line with an angle of depression of 75° does not have the same slope as the line in Problem 1 because an angle of depression falls to the right whereas an angle of elevation rises to the right. The slope of a line that falls to the right is equal to the opposite of the tangent of the angle of depression.



Lines that fall to the right have negative slopes. The negative comes from vertical change in the negative direction. The sides of a triangle have distances greater than zero, so the value of the tangent of an acute angle will be always be positive.

3. Given scalene triangle ABD, C is on $\overline{AB}$, and $DC = 10$, find the perimeter of triangle ABD to the nearest tenth of a unit.



$\tan 80 = \frac{10}{AC}$

$AC = \frac{10}{\tan 80} \approx 1.8$

$\tan 30 = \frac{10}{BC}$

$BC = \frac{10}{\tan 30} \approx 17.3$

Even though the given triangle ABD is not a right triangle, it is composed of two right triangles that share a leg with length 10. Since I know the measures of angles A and B, I can use trigonometric ratios to find the unknown lengths AB, BD, and AD.

$\sin 80 = \frac{10}{AD}$

$AD = \frac{10}{\sin 80} \approx 10.2$

$\sin 30 = \frac{10}{BD}$

$BD = \frac{10}{\sin 30} = \frac{10}{\frac{1}{2}} = 20$

$\text{Perimeter}(ABD) = \left(\frac{10}{\tan 80} + \frac{10}{\tan 30}\right) + \frac{10}{\sin 80} + \frac{10}{\sin 30}$

$\text{Perimeter}(ABD) \approx 49.2$

The perimeter of triangle ABD is approximately 49.2 units.

Lesson 30: Trigonometry and the Pythagorean Theorem

1. If $\cos\alpha = \frac{12\sqrt{3}}{24}$,

   a. Find $\sin\alpha$.

   $$\sin^2\alpha + \cos^2\alpha = 1$$
   $$\sin^2\alpha = 1 - \left(\frac{12\sqrt{3}}{24}\right)^2$$
   $$\sin^2\alpha = 1 - \left(\frac{\sqrt{3}}{2}\right)^2$$
   $$\sin^2\alpha = 1 - \frac{3}{4}$$
   $$\sin^2\alpha = \frac{1}{4}$$
   $$\sin\alpha = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

   If $12\sqrt{3}$ and 24 are considered to be the lengths of the adjacent leg and hypotenuse of a right triangle, respectively, I do not know the length of the opposite leg. I could calculate the length using the Pythagorean theorem, or I could use the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$.

   The notation $\sin^2\theta$ is regularly used as an equivalent, but less cumbersome form, of the expression $(\sin\theta)^2$, and likewise, $\cos^2\theta = (\cos\theta)^2$.

   If $\cos\alpha = \frac{12\sqrt{3}}{24}$, *then* $\sin\alpha = \frac{1}{2}$.

   b. Find $\tan\alpha$.

   $$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$
   $$\tan\alpha = \frac{\frac{1}{2}}{\frac{12\sqrt{3}}{24}}$$
   $$\tan\alpha = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

   If $\cos\alpha = \frac{12\sqrt{3}}{24}$, *then* $\tan\alpha = \frac{1}{\sqrt{3}}$.

2. If $\tan\beta = 3$,

3 is a whole number but can be written as a fraction greater than one. This will provide me with lengths to use for the opposite and adjacent legs of a right triangle.

   a. Find $\sin\beta$.

   $\tan\beta = \frac{3}{1}$

   $3^2 + 1^2 = hyp^2$ *Pythagorean theorem*

   $10 = hyp^2$

   $\sqrt{10} = hyp$

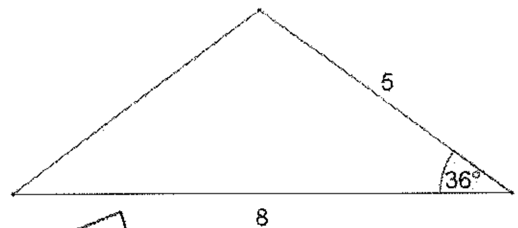
   $\sin\beta = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$

   b. Find $\cos\beta$.

   $\cos\beta = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$

Lesson 31: Using Trigonometry to Determine Area

1. Find the area of the triangle. Round your answer to the nearest tenth.



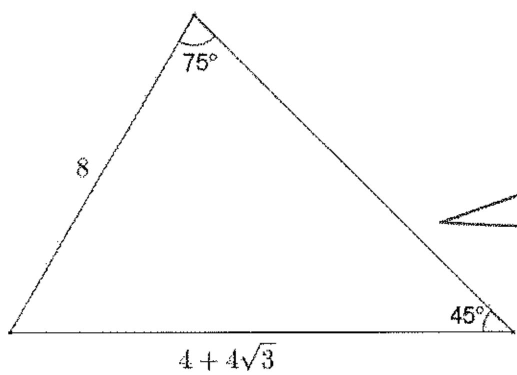
$\text{Area} = \frac{1}{2}(8)(5)(\sin 36)$

$\text{Area} = 20(\sin 36) \approx 11.8$

The area of the triangle is approximately 11.8 *square units.*

> I do not know the height of the triangle, so I cannot use the basic area formula for a triangle. I am given the lengths of two sides of the triangle and the angle between those sides, so I can use the trigonometric area formula.

2. Find the exact area of the triangle.



> I am given the lengths of two sides of the triangle, but I do not have the measure of the included angle. I will have to use the interior angle sum of a triangle to find the measure of the unknown angle first. Then I can use the trigonometric area formula.

$75 + 45 + x = 180$ — *Angle sum of a triangle*

$x = 60$ — *The unknown angle measures* $60°$.

$\text{Area} = \frac{1}{2}(8)(4 + 4\sqrt{3})(\sin 60)$ — *Trigonometric area formula*

$\text{Area} = 4(4 + 4\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)$

$\text{Area} = 4\left(\frac{\sqrt{3}}{2}\right)(4 + 4\sqrt{3})$ — *Associative and commutative properties of multiplication*

$\text{Area} = 2(\sqrt{3})(4 + 4\sqrt{3})$

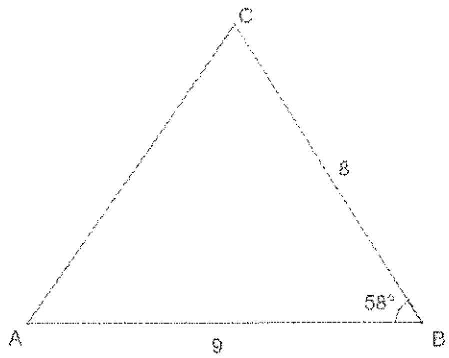
$\text{Area} = 8\sqrt{3} + 8\sqrt{3}\sqrt{3}$

$\text{Area} = 8\sqrt{3} + 8(3)$

$\text{Area} = 8\sqrt{3} + 24$ — *The exact area of the triangle is* $(24 + 8\sqrt{3})$ *square units.*

3. Given $\triangle ABC$, $AB = 9$, $BC = 8$, and $m\angle B = 58°$, find the area of $\triangle ABC$ to the nearest tenth of a square unit.

I should draw a diagram using the given information to help me understand what I can use to find the area of the triangle.



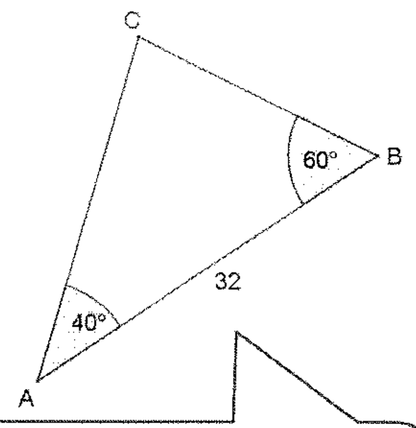
$\textbf{Area} = \frac{1}{2}(9)(8)(\sin 58)$

$\textbf{Area} = 36(\sin 58) \approx 30.5$ ***The area of $\triangle ABC$ is approximately 30.5 square units.***

Lesson 32: Using Trigonometry to Find Side Lengths of an Acute Triangle

Supplemental Problems

1. Given $\triangle ABC$, $AB = 32$, $m\angle A = 40°$, and $m\angle B = 60°$, find $m\angle C$, and use the law of sines to find AC and BC to the nearest tenth.



The problem gives me two angles and the length of one side. I can find the unknown angle using the angle sum of a triangle. Then I can use the law of sines to find the lengths of the two unknown sides.

$40° + 60° + m\angle C = 180°$ *Angle sum of a triangle*

$$m\angle C = 80°$$

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ *Law of sines*

$$\frac{\sin 40}{BC} = \frac{\sin 60}{AC} = \frac{\sin 80}{32}$$

$$\frac{\sin 40}{BC} = \frac{\sin 80}{32}$$

$$BC\left(\frac{\sin 40}{BC}\right) \cdot 32 = BC\left(\frac{\sin 80}{32}\right) \cdot 32$$

$$32 \cdot \sin 40 = BC(\sin 80)$$

$$\frac{1}{\sin 80} \cdot 32 \cdot \sin 40 = \frac{1}{\sin 80} \cdot BC(\sin 80)$$

$$32 \cdot \frac{\sin 40}{\sin 80} = BC$$

$$BC \approx 20.9$$

BC is approximately 20.9 units long.

$$\frac{\sin 60}{AC} = \frac{\sin 80}{32}$$

$$AC\left(\frac{\sin 60}{AC}\right) \cdot 32 = AC\left(\frac{\sin 80}{32}\right) \cdot 32$$

$$32 \cdot \sin 60 = AC(\sin 80)$$

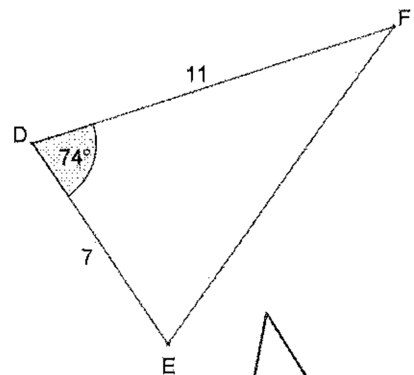
$$\frac{1}{\sin 80} \cdot 32 \cdot \sin 60 = \frac{1}{\sin 80} \cdot AC(\sin 80)$$

$$32 \cdot \frac{\sin 60}{\sin 80} = AC$$

$$AC \approx 28.1$$

AC is approximately 28.1 *units long.*

2. Given $\triangle DEF$, $m\angle D = 74°$, $DF = 11$, and $DE = 7$, use the law of cosines to find EF to the nearest tenth.



$$EF^2 = 7^2 + 11^2 - 2(7)(11)(\cos 74)$$

$$EF^2 = 49 + 121 - 154(\cos 74)$$

$$EF^2 = 170 - 154(\cos 74)$$

$$EF = \sqrt{170 - 154(\cos 74)}$$

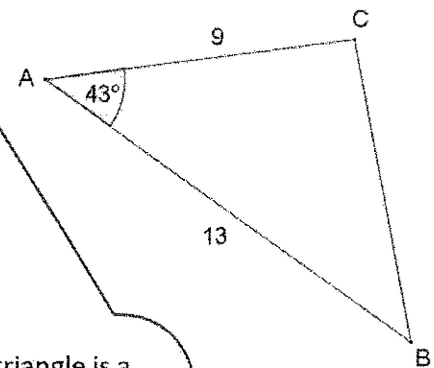
$$EF \approx 11.3$$

EF is approximately 11.3 *units long.*

I don't know the length of the side opposite the only given angle, so I cannot use the law of sines. I do have the lengths of two sides and their included angle, so I can use the law of cosines to find the unknown side length.

Lesson 33: Applying the Laws of Sines and Cosines

1. Given $\triangle ABC$, $AC = 9$, $AB = 13$, and $m\angle A = 43°$, find BC to the nearest tenth. Explain your strategy.



   Two side lengths of the triangle and the included angle measure are known, so I must use the law of cosines to find BC.

   $BC^2 = 9^2 + 13^2 - 2(9)(13)(\cos 43)$

   $BC^2 = 81 + 169 - 234(\cos 43)$

   $BC^2 = 250 - 234(\cos 43)$

   $BC = \sqrt{250 - 234(\cos 43)}$

   $BC \approx 8.9$

   BC is approximately 8.9 units long.

   I do not know if the triangle is a right triangle or not, so I cannot use the Pythagorean theorem. I do not know the length of the side opposite angle A, so I cannot use the law of sines. I am given the lengths of two sides and their included angle, which will allow me to use the law of cosines to find BC.

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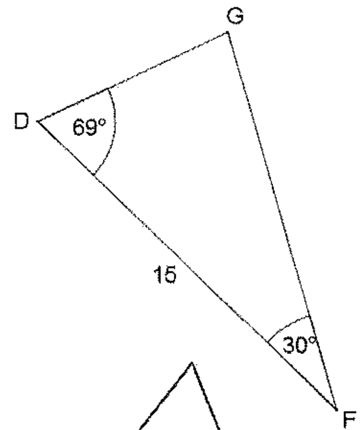
2. Given $\triangle DFG$, $DF = 15$, $m\angle D = 69°$, and $m\angle F = 30°$, find DG and FG.



$$69° + 30° + m\angle G = 180°$$
$$m\angle G = 81°$$

If the angles in a triangle as well as a side length opposite one of those angles are known, the law of sines can be used to find the lengths of the remaining sides.

If I am given two angle measures of a triangle, I know the sum of all three angles is 180°, so the third angle is easy to find. I have the measures of the angles in a triangle and at least one side length, so I can find the remaining side lengths using the law of sines.

$$\frac{\sin D}{d} = \frac{\sin F}{f} = \frac{\sin G}{g}$$

$$\frac{\sin 69}{FG} = \frac{\sin 81}{15}$$
$$FG = 15 \cdot \frac{\sin 69}{\sin 81} \approx 14.2$$

FG is approximately 14.2 units long.

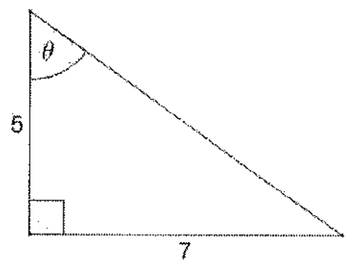
$$\frac{\sin 30}{DG} = \frac{\sin 81}{15}$$
$$DG = 15 \cdot \frac{\sin 30}{\sin 81} \approx 7.6$$

DG is approximately 7.6 units long.

Lesson 34: Unknown Angles

1. For each triangle, find the measure of the unknown angle θ to the nearest tenth of a degree.

   a.



I am given the lengths of both legs of the right triangle, so I can use arctan to find the missing angle measure without having to do any other calculations.

$$\tan\theta = \frac{7}{5}$$

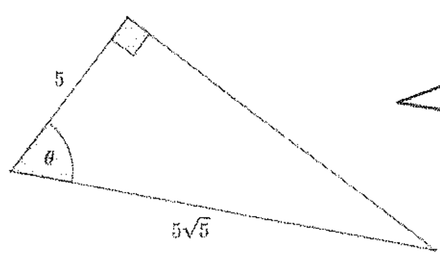
$$\arctan(\tan\theta) = \arctan\left(\frac{7}{5}\right)$$

$$\theta \approx 54.5$$

The measure of the unknown angle is approximately $54.5°$.

Most calculators display the notation $\sin^{-1} x$ to represent arcsin (and similarly so for arccos and arctan).

   b.



I am given the lengths of the leg that is adjacent to the unknown angle and the hypotenuse, so I can use arccos to find the missing angle measure without having to do any other calculations.

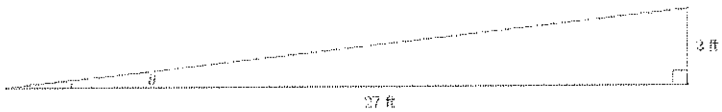
$$\cos\theta = \frac{5}{5\sqrt{5}}$$

$$\arccos(\cos\theta) = \arccos\left(\frac{5}{5\sqrt{5}}\right)$$

$$\theta \approx 63.4$$

The measure of the unknown angle is approximately $63.4°$.

2. A ramp is built so that it has a vertical rise of 3 ft. over a horizontal distance of 27 ft. Find the approximate angle of elevation that the ramp makes with the ground (assume the ground is a horizontal grade).



I should draw a diagram as described by the problem to help me visualize the problem.

$$\tan\theta = \frac{3}{27}$$

$$\arctan(\tan\theta) = \arctan\left(\frac{3}{27}\right)$$

$$\theta \approx 6.3$$

The angle of elevation of the ramp is approximately $6.3°$ from the level ground.

My diagram reveals known lengths of both legs of a right triangle. That means I can use arctan to calculate the unknown angle measure θ.