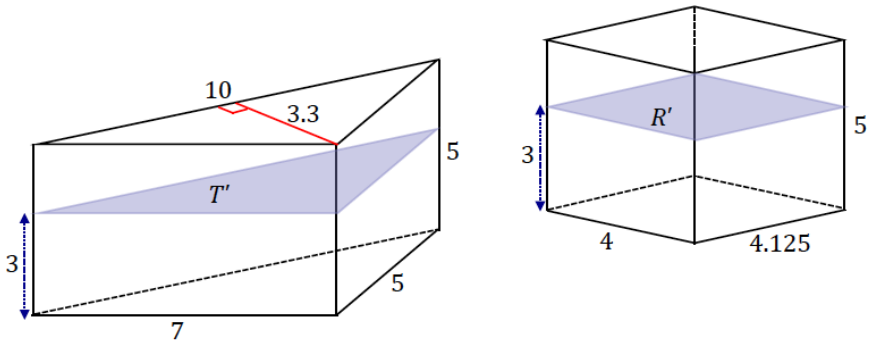
Lesson 10: The Volume of Prisms and Cylinders and Cavalieri's Principle

Classwork

Opening Exercise

The bases of the following triangular prism T and rectangular prism R lie in the same plane. A plane that is parallel to the bases and also a distance 3 from the bottom base intersects both solids and creates cross-sections T' and R'.



a. Find $\text{Area}(T')$.

b. Find $\text{Area}(R')$.

c. Find $\text{Vol}(T)$.

d. Find $\text{Vol}(R)$.

e. If a height other than 3 were chosen for the cross-section, would the cross-sectional area of either solid change?

Discussion

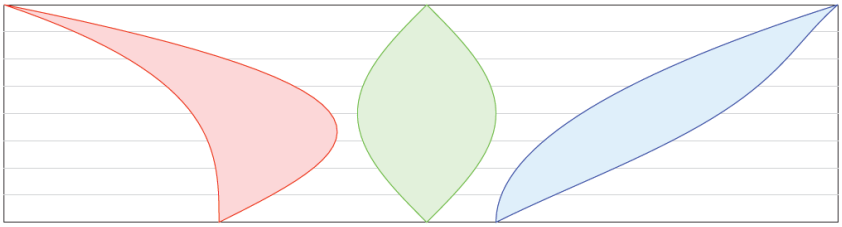
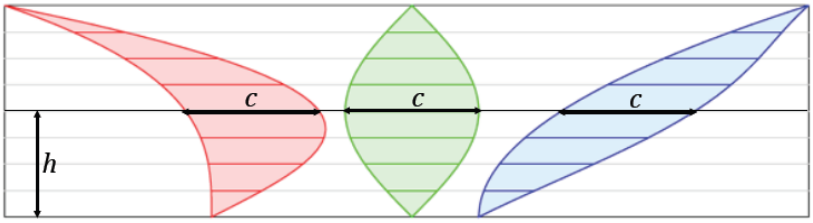
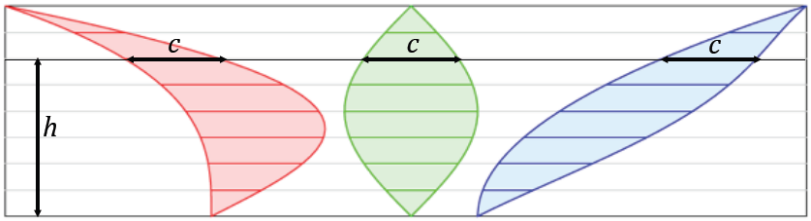


Figure 1

Example 1



Example 2



PRINCIPLE OF PARALLEL SLICES IN THE PLANE: If two planar figures of equal altitude have identical cross-sectional lengths at each height, then the regions of the figures have the same area.

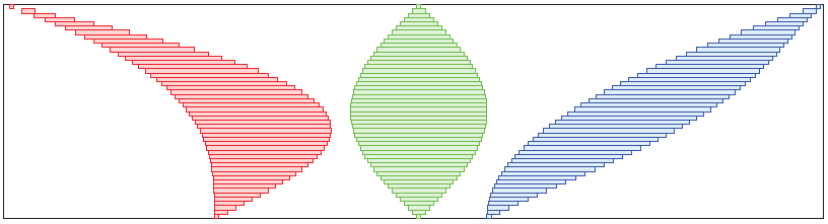
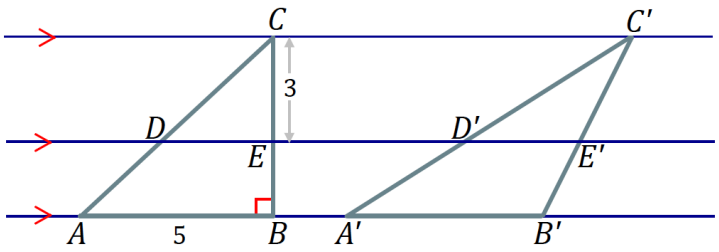


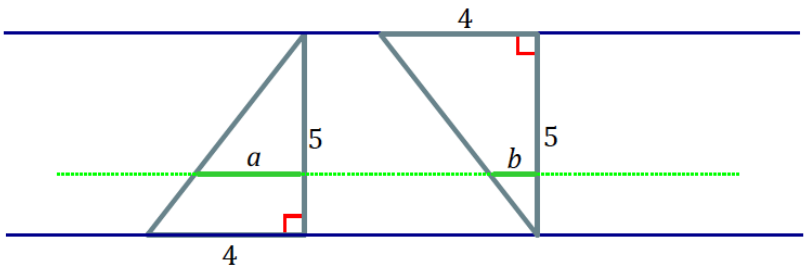
Figure 2

Example

a. The following triangles have equal areas: $\text{Area}(\triangle ABC) = \text{Area}(\triangle A'B'C') = 15$ units^2. The distance between $\overleftrightarrow{DE}$ and $\overleftrightarrow{CC'}$ is 3. Find the lengths of $\overline{DE}$ and $\overline{D'E'}$.



b. Joey says that if two figures have the same height and the same area, then their cross-sectional lengths at each height will be the same. Give an example to show that Joey's theory is incorrect.



Discussion

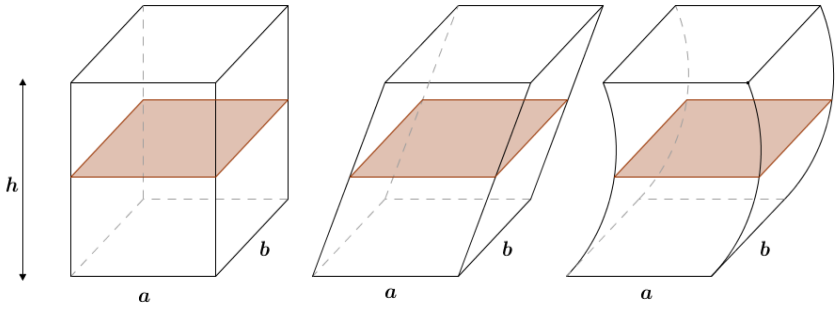


Figure 3

CAVALIERI'S PRINCIPLE: Given two solids that are included between two parallel planes, if every plane parallel to the two planes intersects both solids in cross-sections of equal area, then the volumes of the two solids are equal.

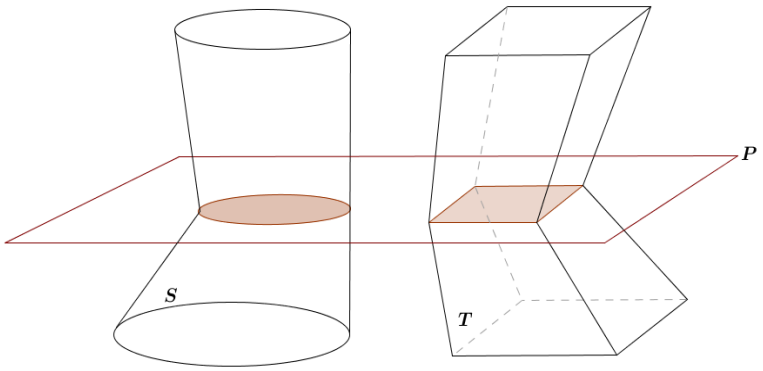


Figure 4

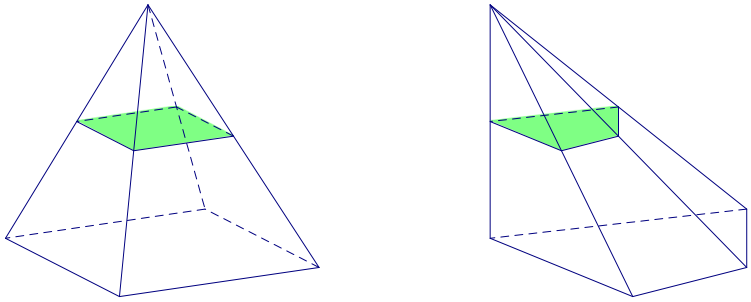


Figure 5

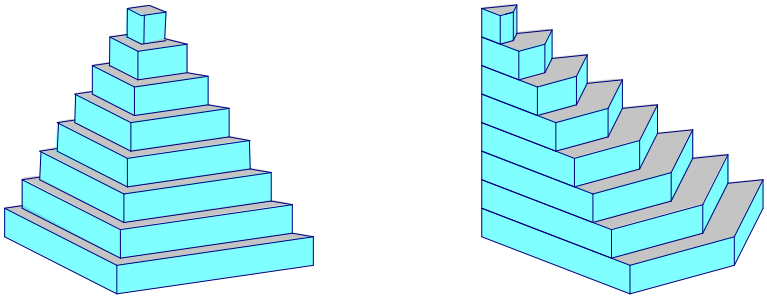


Figure 6

Lesson Summary

PRINCIPLE OF PARALLEL SLICES IN THE PLANE: If two planar figures of equal altitude have identical cross-sectional lengths at each height, then the regions of the figures have the same area.

CAVALIERI'S PRINCIPLE: Given two solids that are included between two parallel planes, if every plane parallel to the two planes intersects both solids in cross-sections of equal area, then the volumes of the two solids are equal.

Problem Set

1. Use the principle of parallel slices to explain the area formula for a parallelogram.

2. Use the principle of parallel slices to show that the three triangles shown below all have the same area.

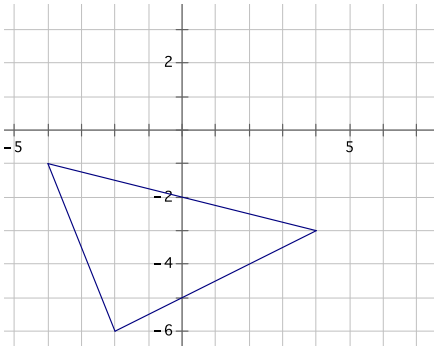


Figure 1

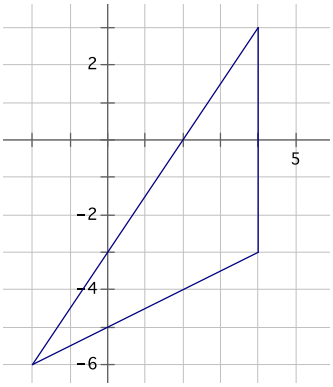


Figure 2

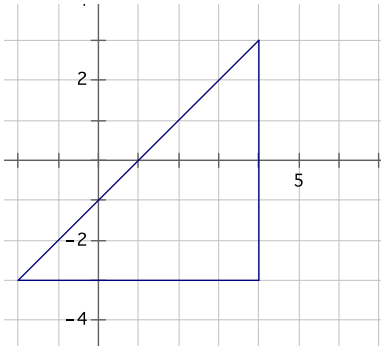
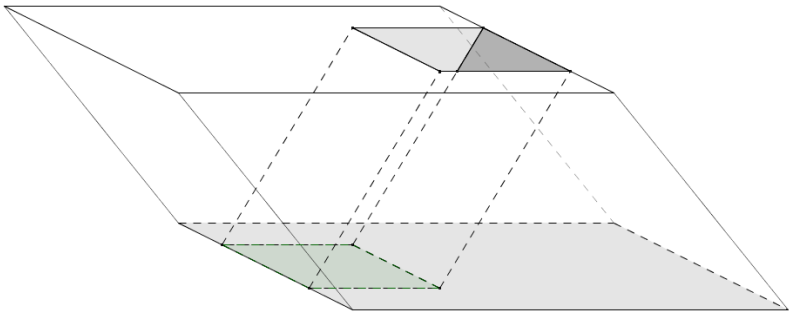
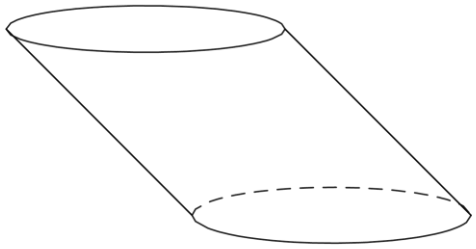


Figure 3

3. An oblique prism has a rectangular base that is 16 in. $\times$ 9 in. A hole in the prism is also the shape of an oblique prism with a rectangular base that is 3 in. wide and 6 in. long, and the prism's height is 9 in. (as shown in the diagram). Find the volume of the remaining solid.



4. An oblique circular cylinder has height 5 and volume 45π. Find the radius of the circular base.



5. A right circular cone and a solid hemisphere share the same base. The vertex of the cone lies on the hemisphere. Removing the cone from the solid hemisphere forms a solid. Draw a picture, and describe the cross-sections of this solid that are parallel to the base.

6. Use Cavalieri's principle to explain why a circular cylinder with a base of radius 5 and a height of 10 has the same volume as a square prism whose base is a square with edge length $5\sqrt{\pi}$ and whose height is also 10.