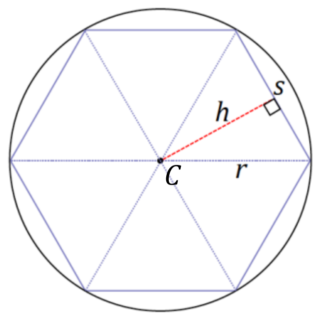
Lesson 4: Proving the Area of a Disk

Classwork

Opening Exercise

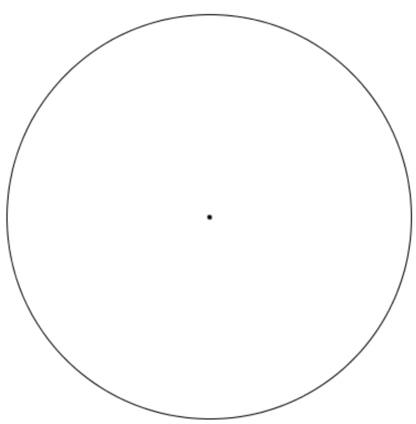
The following image is of a regular hexagon inscribed in circle C with radius r. Find a formula for the area of the hexagon in terms of the length of a side, s, and the distance from the center to a side.



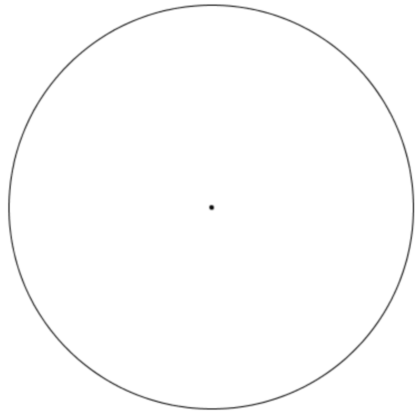
Example

a. Begin to approximate the area of a circle using inscribed polygons.

How well does a square approximate the area of a disk? Create a sketch of P_4 (a regular polygon with 4 sides, a square) in the following circle. Shade in the area of the disk that is not included in P_4.



b. Next, create a sketch of P_8 in the following circle.



c. Indicate which polygon has a greater area.

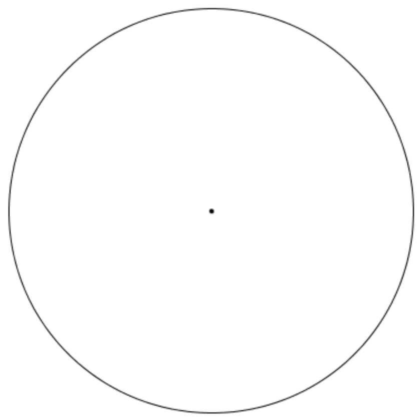
$$\text{Area}(P_4) \text{ \_\_\_\_\_\_\_\_\_\_ } \text{Area}(P_8)$$

d. Will the area of inscribed regular polygon P_{16} be greater or less than the area of P_8? Which is a better approximation of the area of the disk?

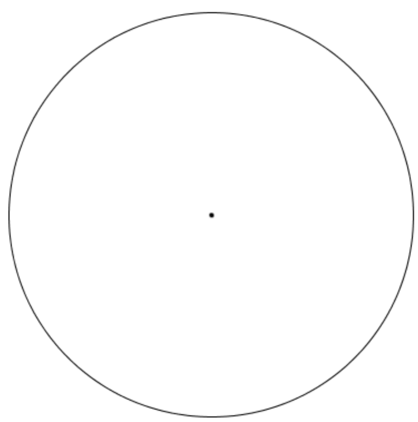
e. We noticed that the area of P_4 was less than the area of P_8 and that the area of P_8 was less than the area of P_{16}. In other words, $\text{Area}(P_n)$ ________ $\text{Area}(P_{2n})$. Why is this true?

f. Now we will approximate the area of a disk using circumscribed (outer) polygons.

Now circumscribe, or draw a square on the outside of, the following circle such that each side of the square intersects the circle at one point. We will denote each of our outer polygons with prime notation; we are sketching P'_4 here.



g. Create a sketch of P'_8.



h. Indicate which polygon has a greater area.

$$\text{Area}(P'_4) \underline{\hspace{3cm}} \text{Area}(P'_8)$$

i. Which is a better approximation of the area of the circle, P'_4 or P'_8? Explain why.

j. How will $\text{Area}(P'_n)$ compare to $\text{Area}(P'_{2n})$? Explain.

LIMIT (description): Given an infinite sequence of numbers, $a_1, a_2, a_3, \ldots$, to say that *the limit of the sequence is* A means, roughly speaking, that when the index n is very large, then a_n is very close to A. This is often denoted as, "As $n \to \infty$, $a_n \to A$."

AREA OF A CIRCLE (description): The *area of a circle* is the limit of the areas of the inscribed regular polygons as the number of sides of the polygons approaches infinity.

Problem Set

1. Describe a method for obtaining closer approximations of the area of a circle. Draw a diagram to aid in your explanation.

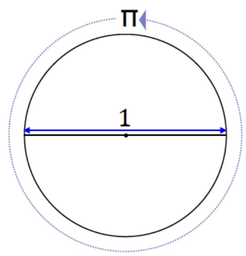
2. What is the radius of a circle whose circumference is π?

3. The side of a square is 20 cm long. What is the circumference of the circle when …
   a. The circle is inscribed within the square?
   b. The square is inscribed within the circle?

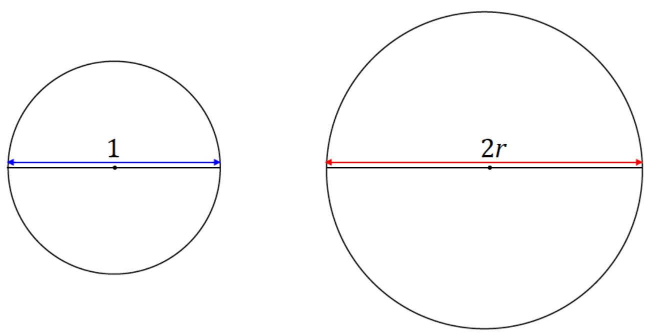
4. The circumference of circle C_1 is 9 cm, and the circumference of C_2 is $2\pi\text{ cm}$. What is the value of the ratio of the areas of C_1 to C_2?

5. The circumference of a circle and the perimeter of a square are each 50 cm. Which figure has the greater area?

6. Let us define π to be the circumference of a circle whose diameter is 1.



We are going to show why the circumference of a circle has the formula $2\pi r$. Circle C_1 below has a diameter of $d = 1$, and circle C_2 has a diameter of $d = 2r$.



   a. All circles are similar (proved in Module 2). What scale factor of the similarity transformation takes C_1 to C_2?
   b. Since the circumference of a circle is a one-dimensional measurement, the value of the ratio of two circumferences is equal to the value of the ratio of their respective diameters. Rewrite the following equation by filling in the appropriate values for the diameters of C_1 and C_2:

$$\frac{\text{Circumference}(C_2)}{\text{Circumference}(C_1)} = \frac{\text{diameter}(C_2)}{\text{diameter}(C_1)}.$$

c. Since we have defined π to be the circumference of a circle whose diameter is 1, rewrite the above equation using this definition for C_1.

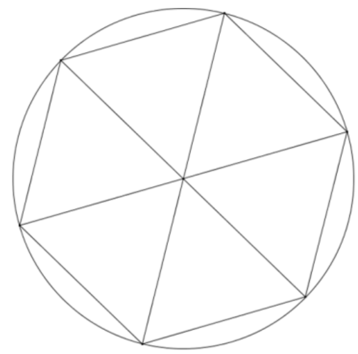
d. Rewrite the equation to show a formula for the circumference of C_2.

e. What can we conclude?

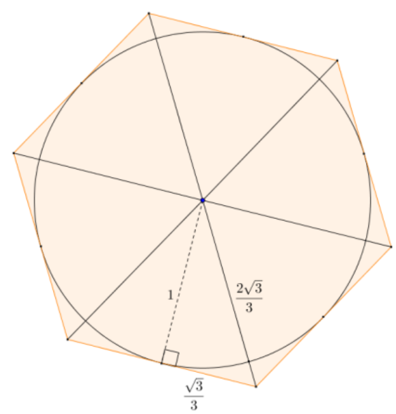
7.

a. Approximate the area of a disk of radius 1 using an inscribed regular hexagon. What is the percent error of the approximation?

(Remember that percent error is the absolute error as a percent of the exact measurement.)



b. Approximate the area of a circle of radius 1 using a circumscribed regular hexagon. What is the percent error of the approximation?



c. Find the average of the approximations for the area of a circle of radius 1 using inscribed and circumscribed regular hexagons. What is the percent error of the average approximation?

8. A regular polygon with n sides each of length s is inscribed in a circle of radius r. The distance h from the center of the circle to one of the sides of the polygon is over 98% of the radius. If the area of the polygonal region is 10, what can you say about the area of the circumscribed regular polygon with n sides?